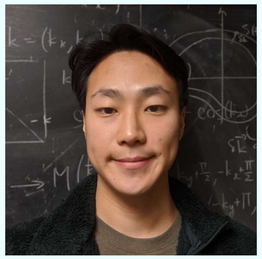
Quantum kinetics, non-linear response, and band geometry

Leon Balents, KITP

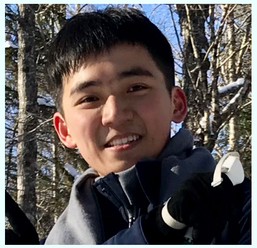
Flat bands: frustration and correlation physics, Sept 2025

Collaborators

Lucile Savary
FACTS



Takamori Park
UCSB -> Princeton



Xiaoyang Huang
CU Boulder -> PI

Wen Wang
UCSB

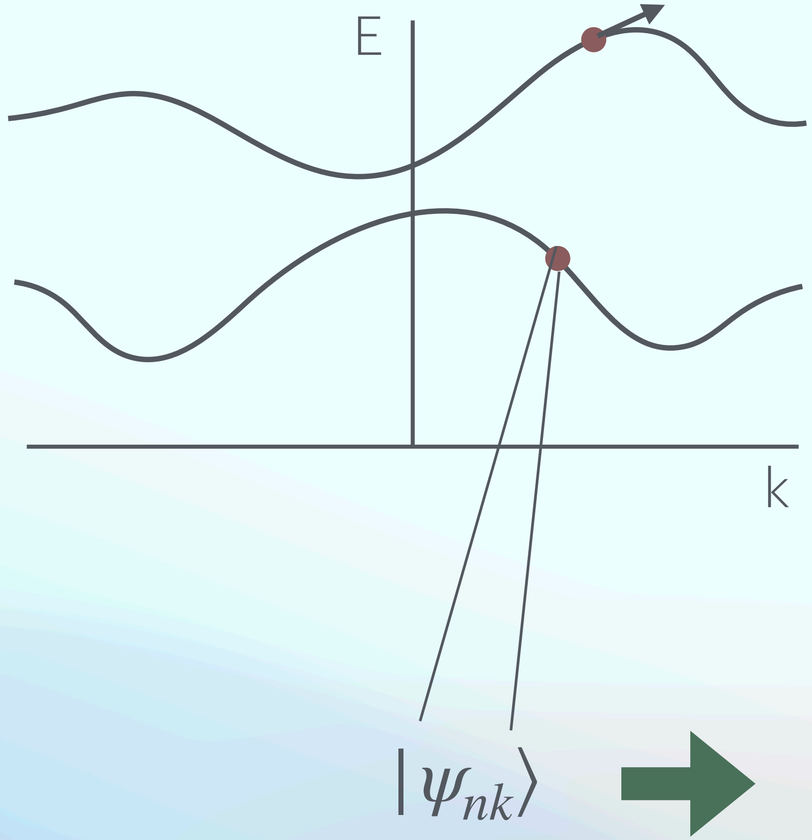
Jia-Xin Zhang
FACTS

Dongning (Donna) Sheng
CSUN

Outline

- Bands and quantum geometry
- Quantum geometric nesting in flat bands
- Doing semiclassics correctly in dispersive bands with quantum geometry
- A quick advertisement for recent work on topological superconductivity

Bands



$\epsilon_n(k)$ → $\mathbf{v}_n(k)$

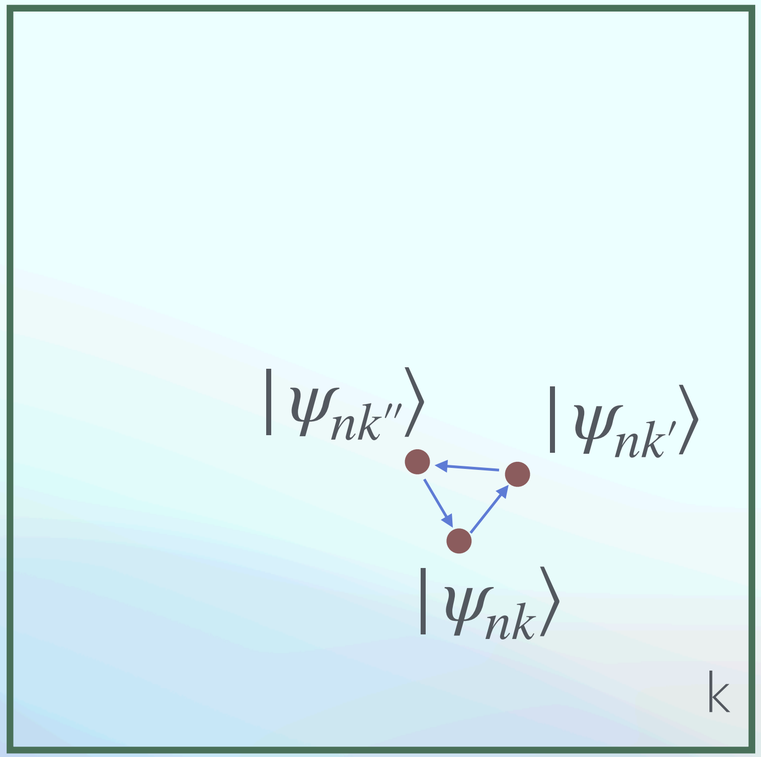
Energetics

Geometry

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Quantum geometry

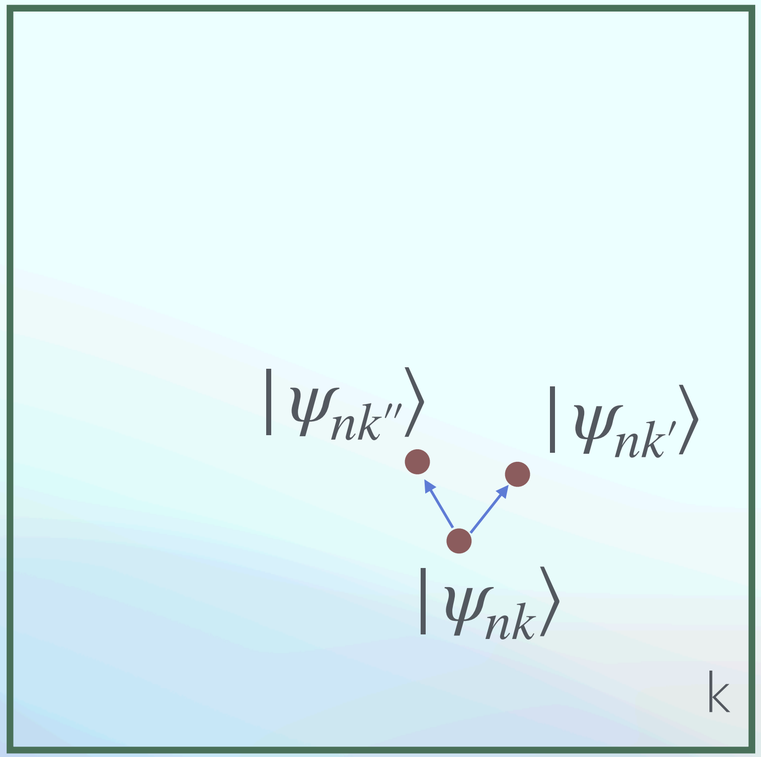


Phase of Bloch state varies with k

Berry phase around a loop is given by the **Berry curvature**.

$$\Omega_{n,\alpha\beta}(k)$$

Quantum geometry

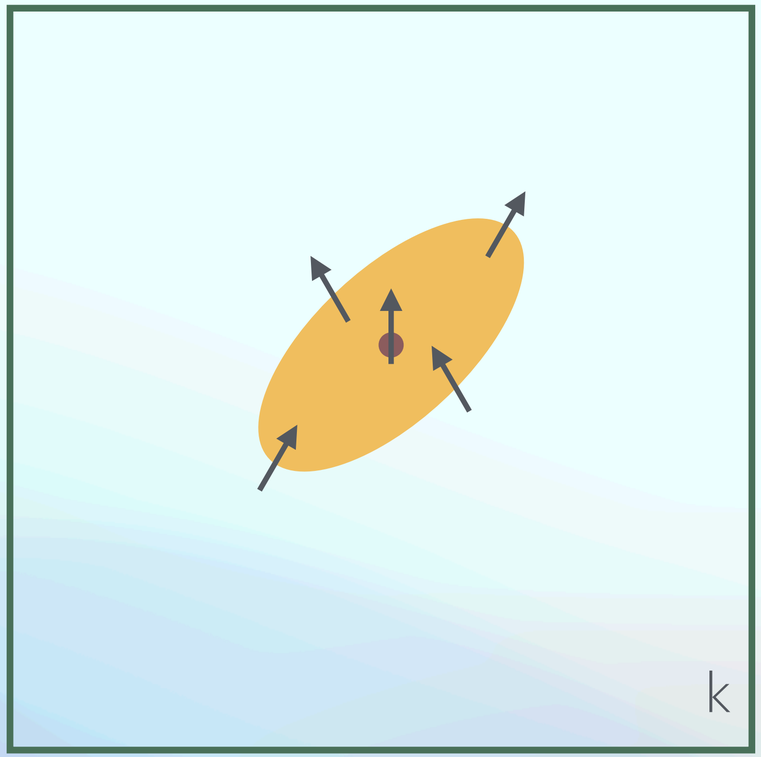


Wavefunction of Bloch state varies with k

How fast they vary is described by **quantum metric**

$$|\langle\psi_{nk}|\psi_{nk+q}\rangle|^2 \sim 1 - g_{\mu\nu}(k)q_\mu q_\nu$$

Quantum geometry

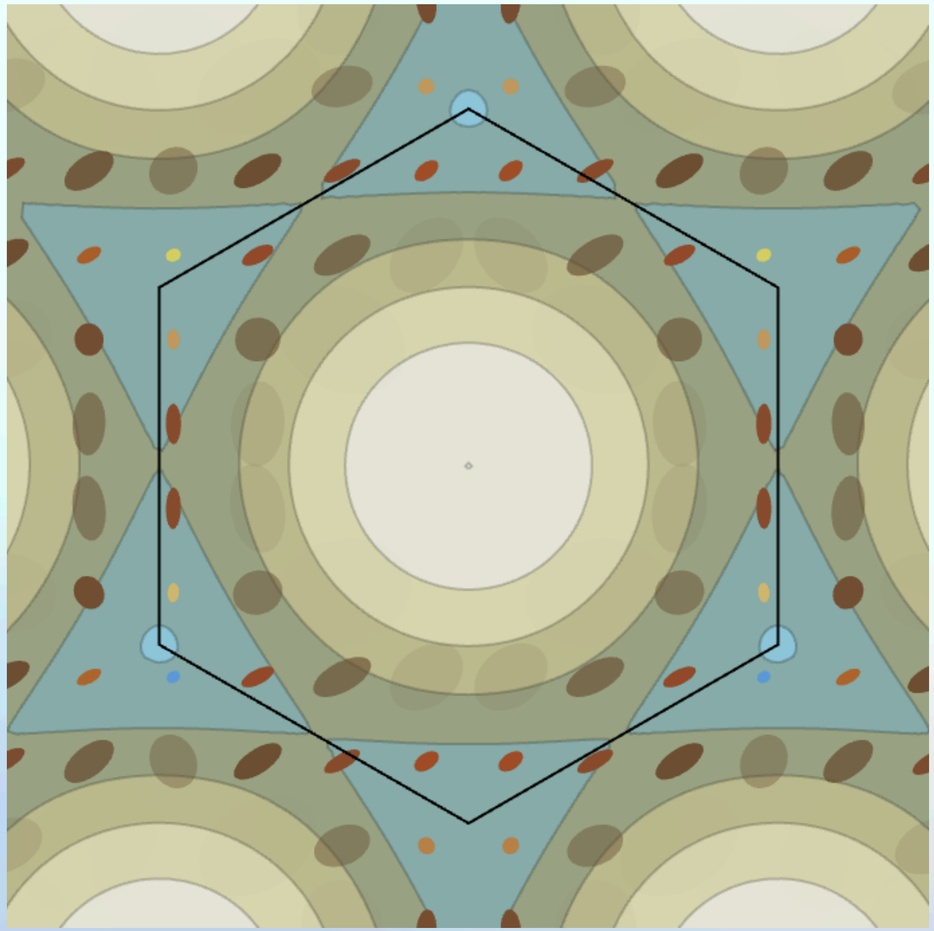


Wavefunction of Bloch state varies with k

How fast they vary is described by **quantum metric**

$$|\langle \psi_{nk} | \psi_{nk+q} \rangle|^2 \sim 1 - g_{\mu\nu}(k) q_\mu q_\nu$$

Quantum geometry



Example: Haldane model

Wavefunction of Bloch state varies with k

How fast they vary is described by **quantum metric**

$$|\langle \psi_{nk} | \psi_{nk+q} \rangle|^2 \sim 1 - g_{\mu\nu}(k) q_\mu q_\nu$$

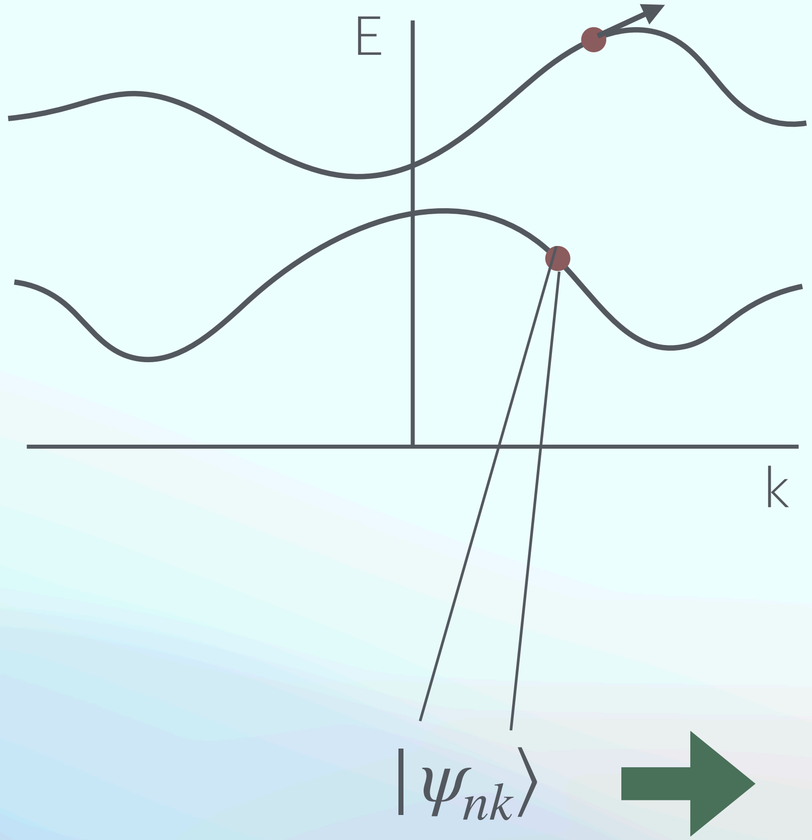
Ellipses show constant “distance”

Quantum geometric nesting

Wen Wang
UCSB

Jia-Xin Zhang
FACTS

Bands



$\epsilon_n(k)$ → $\mathbf{v}_n(k)$

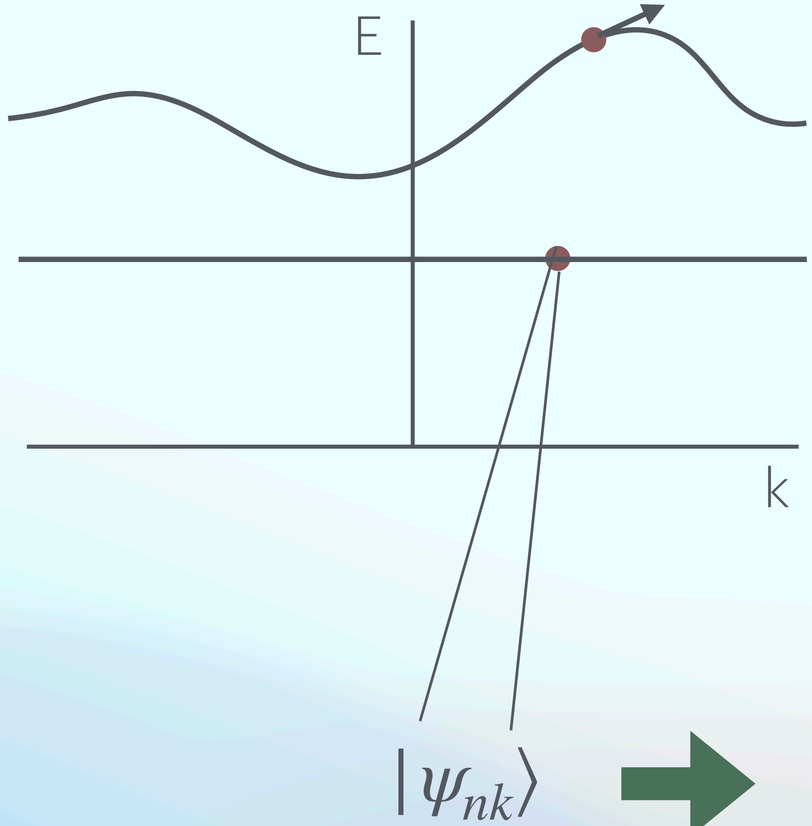
Energetics

Geometry

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Flat bands



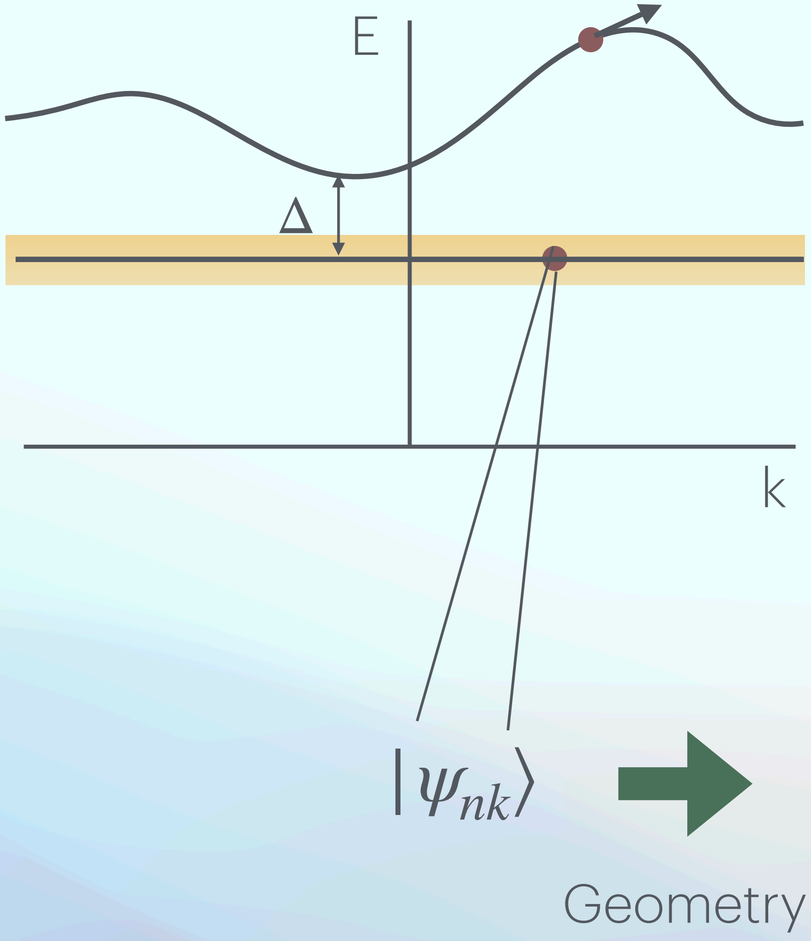
Geometry

In the flat band limit, there is *only* geometry

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Flat bands



In the flat band limit, there is *only* geometry

Interaction $U \ll \Delta$

How does geometry determine the instabilities?

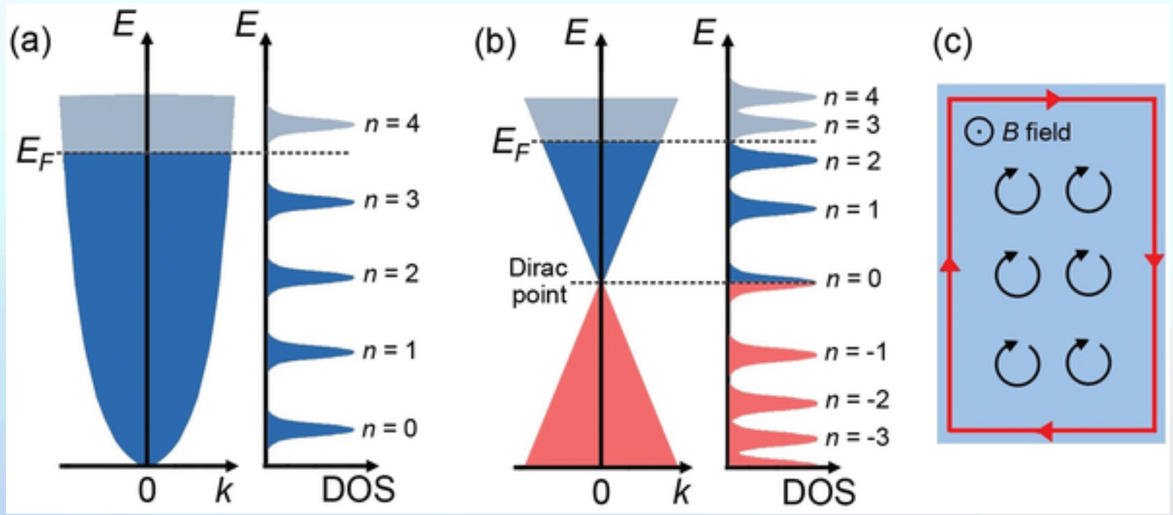
$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

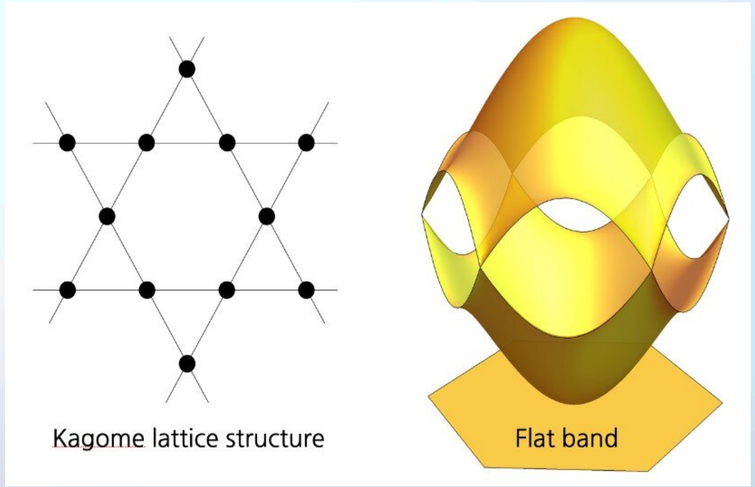
Flat bands

Generally a result of interference in electron motion, which can be induced by frustration or by magnetic flux.

Landau levels

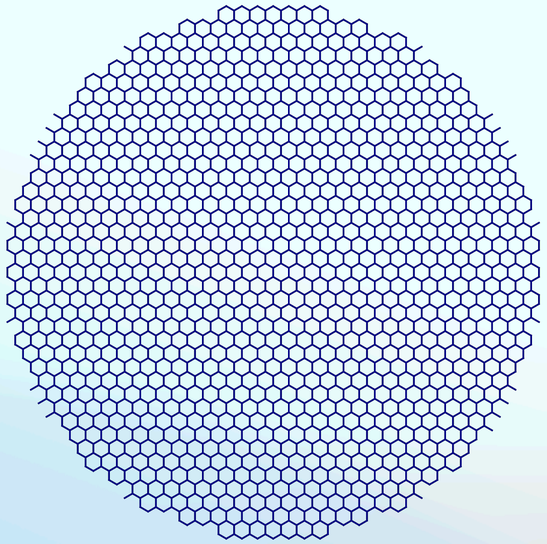


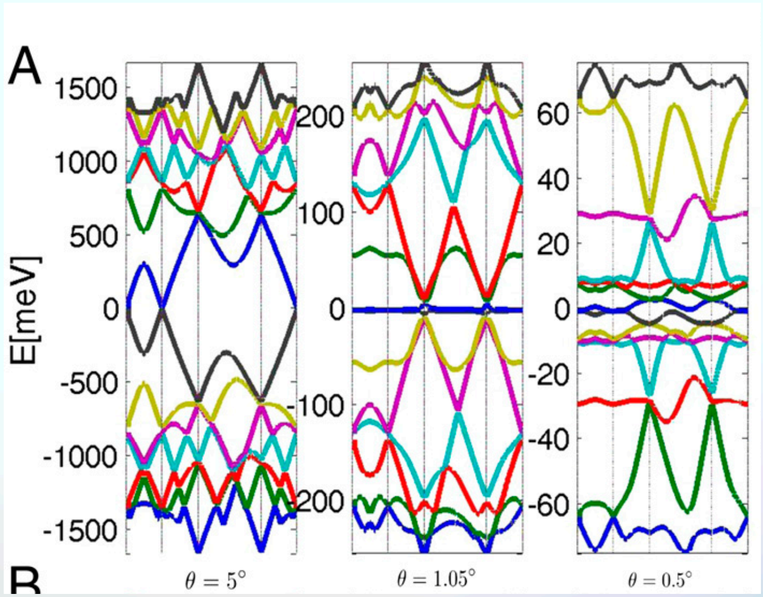
Frustrated hopping



Flat bands

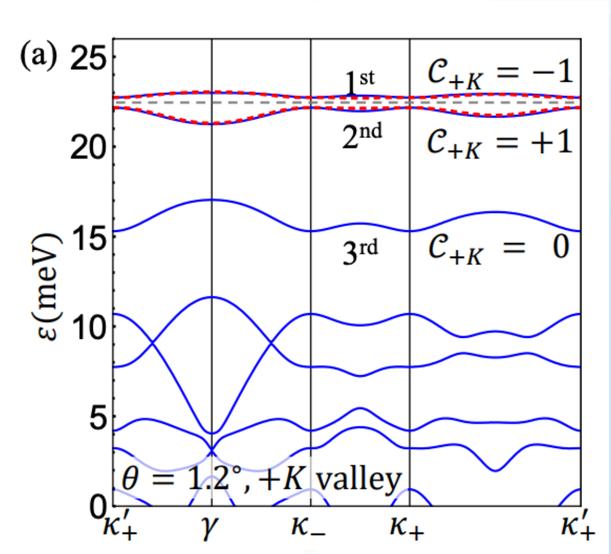
moiré





Bistritzer+Macdonald, 2011

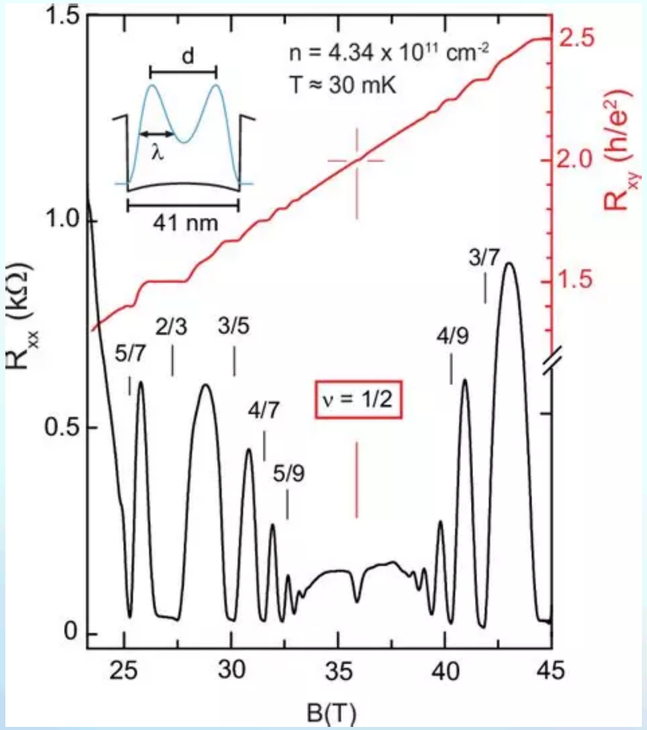
Graphene



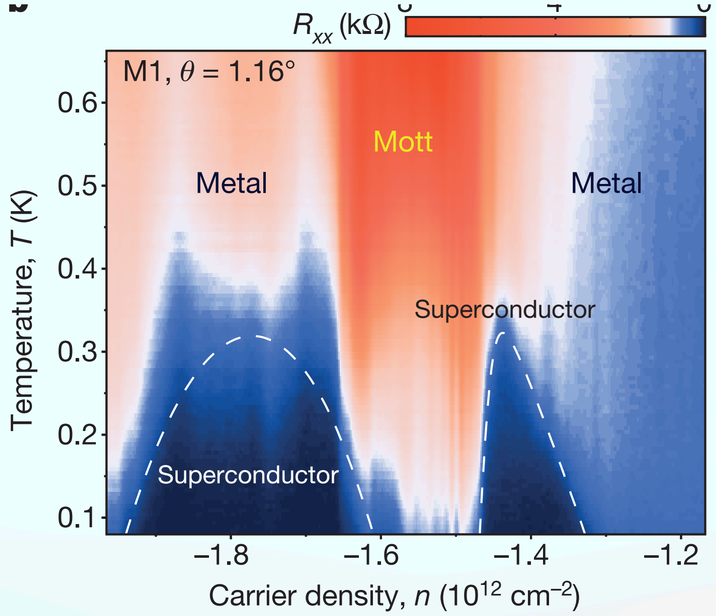
F. Wu *et al*, 2019

TMDs

Correlation physics

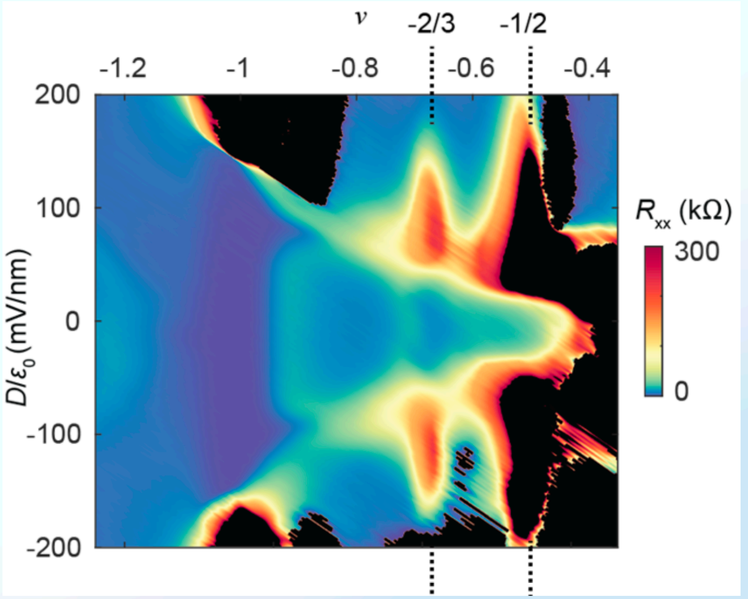


FQHE



Cao *et al*, 2018

Superconductivity and correlated insulators in TBG



H. Park *et al*, 2023

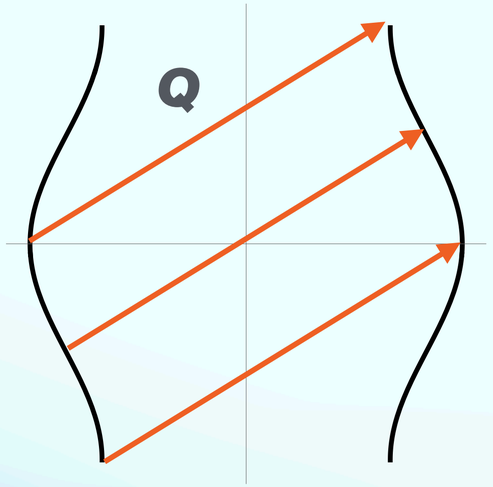
FQAHE in twisted MoTe2

So many different orders

- Ferromagnetism
- Antiferromagnetism
- Intervalley coherence
- Fractional quantum Hall effect
- Stripes
- Charge density wave
- Loop currents
- Superconductivity (s-wave, p-wave, d-wave, FFLO…)

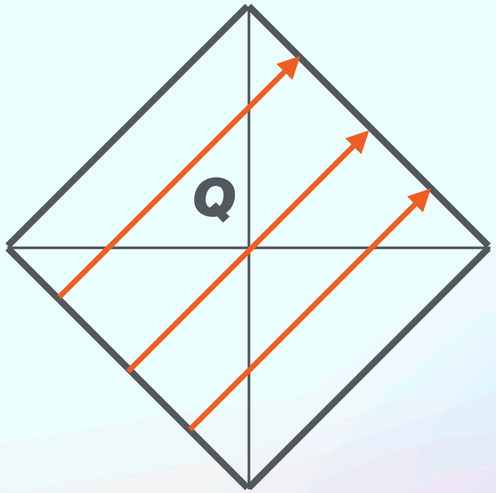
Nesting

Within a *dispersive* band, Fermi surface structure controls instabilities



Quasi-1d

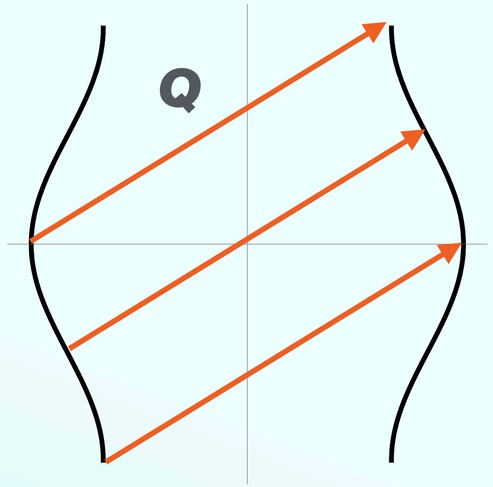
"$2k_F$" CDW/SDW



Square lattice Hubbard

(π, π) antiferromagnet

Susceptibility



$$\chi(Q,\omega) \sim \sum_k \frac{n_F(\epsilon_{k+Q}) - n_F(\epsilon_k)}{\omega - \epsilon_k + \epsilon_{k+Q}}$$

Restricts to near Fermi surface

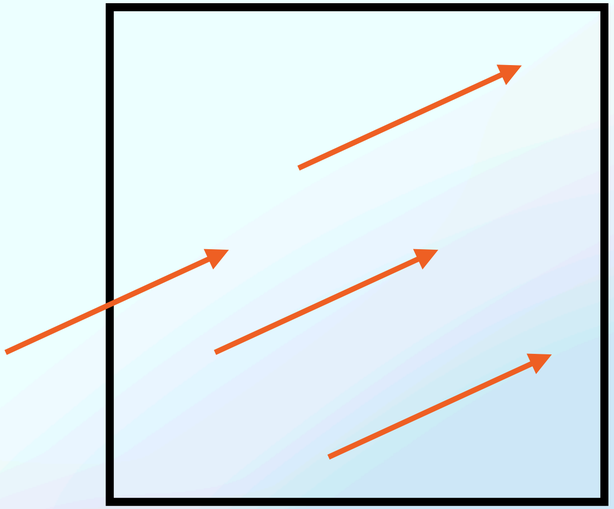
Enhanced susceptibility when this vanishes for extended range of k

What happens when the band is flat?? No Fermi surface!

Quantum geometric nesting

Z. Han *et al*, 2024

For flat band (or bands) at the Fermi energy, susceptibility is rendered finite only by thermal fluctuations. But there is still structure.



$$\chi_{\boldsymbol{Q}} = \frac{1}{T}\sum_{\boldsymbol{k}} \mathrm{Tr}\left[\mathcal{O}^{\dagger}(\boldsymbol{k})P(\boldsymbol{k}+\boldsymbol{Q})\mathcal{O}(\boldsymbol{k})P(\boldsymbol{k})\right].$$

Order parameter

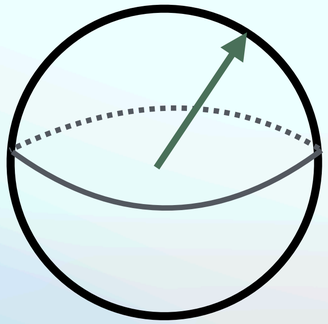
Projector

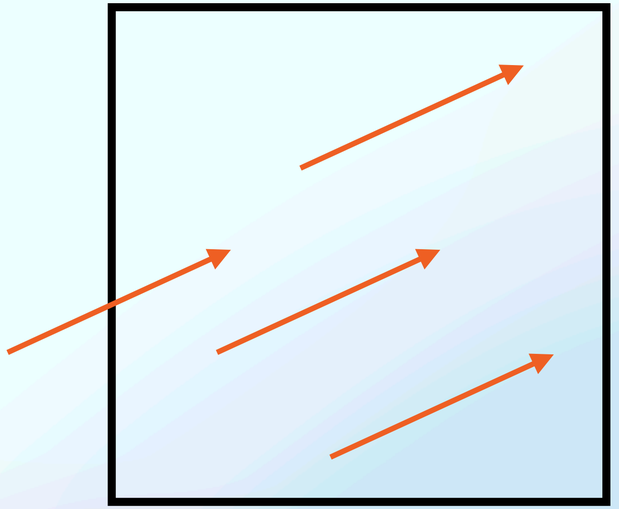
$$P(\boldsymbol{k}) = \sum_{n\in \mathrm{FB}} |\psi_{n\boldsymbol{k}}\rangle\langle\psi_{n\boldsymbol{k}}|$$

Quantum geometric nesting

Bloch vector representation

e.g. 2-band $P(\boldsymbol{k}) = \frac{1}{2} + \frac{1}{2}\boldsymbol{b}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$





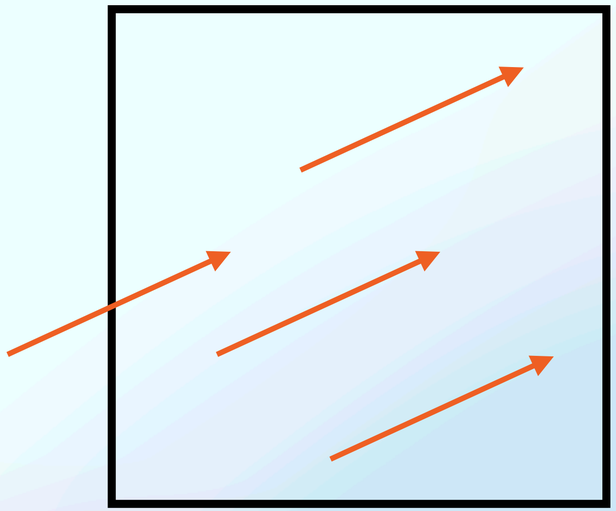
$\mathcal{O}(\boldsymbol{k}) = o(\boldsymbol{k}) + \boldsymbol{o}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$

Quantum geometric nesting

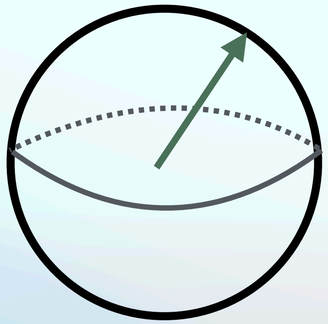
Bloch vector representation

N bands, N_L flat

$$P(\boldsymbol{k}) = \frac{N_L}{N} + \frac{1}{2}\boldsymbol{b}(\boldsymbol{k}) \cdot \boldsymbol{\lambda}$$



$\boldsymbol{\lambda} \in \mathrm{SU(N)}$



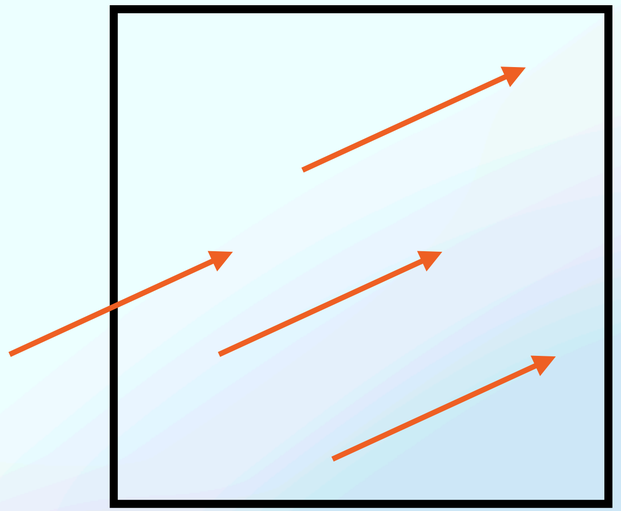
$$\mathcal{O}(\boldsymbol{k}) = o(\boldsymbol{k}) + \boldsymbol{o}(\boldsymbol{k}) \cdot \boldsymbol{\lambda}$$

Quantum geometric nesting

$$\chi(\boldsymbol{Q}) = \sum_{\boldsymbol{k}} \frac{\mathrm{Tr}[\mathcal{O}^\dagger(\boldsymbol{k})\mathcal{O}(\boldsymbol{k})]}{T} \left(\frac{N_L^2}{N} + \frac{1}{2}\tilde{\boldsymbol{b}}_{\boldsymbol{o}}(\boldsymbol{k}+\boldsymbol{Q}) \cdot \boldsymbol{b}(\boldsymbol{k}) \right),$$

Relates quantum geometry at **k** and **k+Q**

$$\tilde{\boldsymbol{b}}_{\boldsymbol{o}}(\boldsymbol{k}+\boldsymbol{Q}) \equiv \boldsymbol{b}(\boldsymbol{k}+\boldsymbol{Q}) - N\left[\hat{\boldsymbol{o}}(\boldsymbol{k}) \times \boldsymbol{b}(\boldsymbol{k}+\boldsymbol{Q}) \times \hat{\boldsymbol{o}}(\boldsymbol{k})\right]$$



QGN condition:

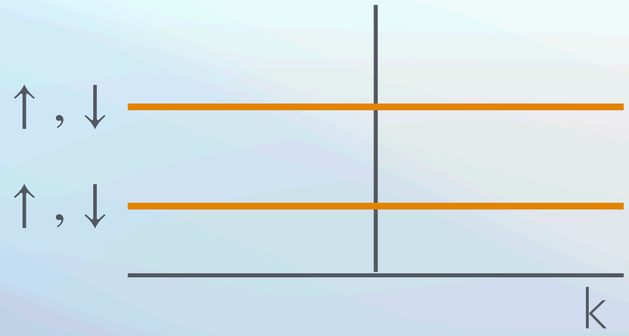
$$\tilde{\boldsymbol{b}}_{\boldsymbol{o}}(\boldsymbol{k}+\boldsymbol{Q}) \quad \| \quad \boldsymbol{b}(\boldsymbol{k}), \qquad \forall \boldsymbol{k} \in \mathrm{BZ}$$

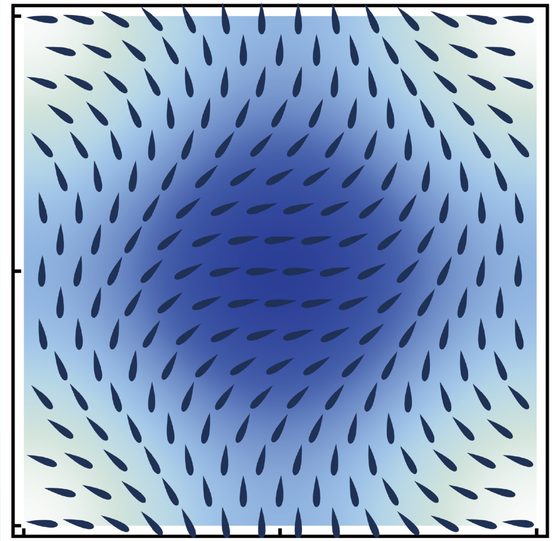
Example

JS Hofmann *et al*, 2022,2023

$$H_0 = t \sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} \begin{pmatrix} -\mu & -ie^{i\alpha_k^\uparrow} & 0 & 0 \\ ie^{-i\alpha_k^\uparrow} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha_k^\downarrow} \\ 0 & 0 & -ie^{i\alpha_k^\downarrow} & -\mu \end{pmatrix} c_{\boldsymbol{k}}$$

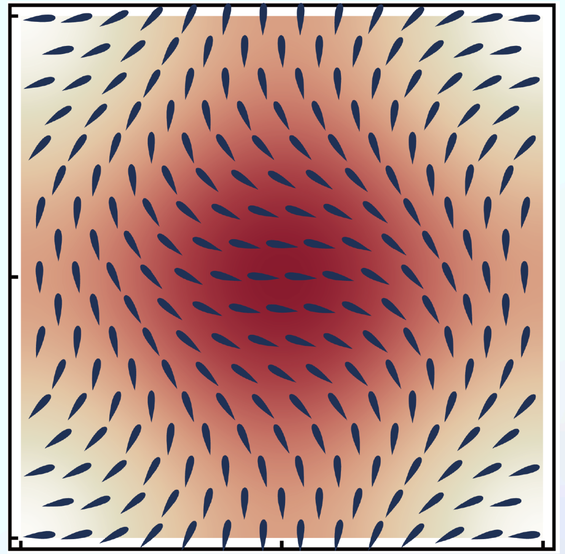
$$\alpha_k^\sigma = \eta_\sigma \left(\cos k_x + \cos k_y\right)$$

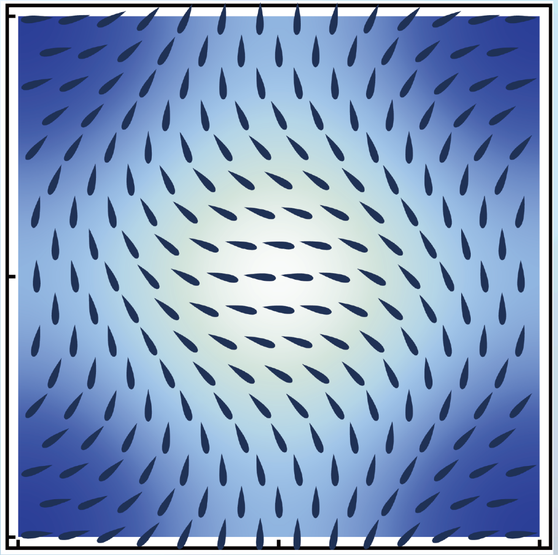




$b(\boldsymbol{k})$

$\tilde{b}_M(\boldsymbol{k})$





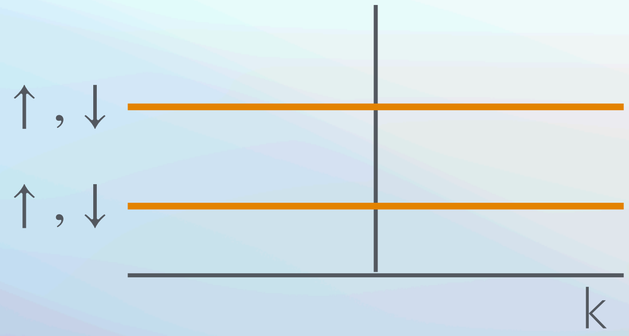
$\tilde{b}_N(\boldsymbol{k})$

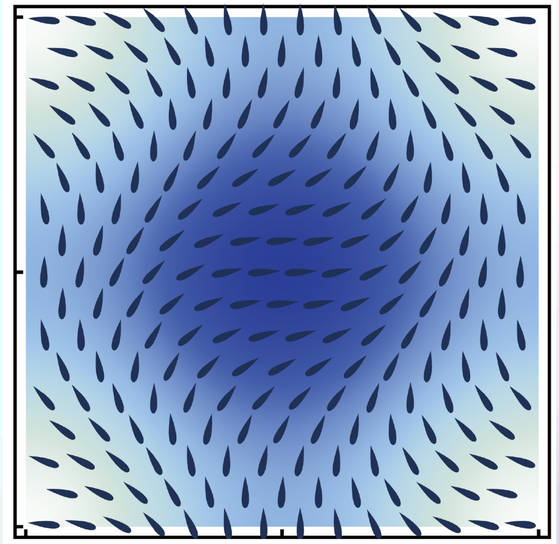
Example

JS Hofmann *et al*, 2022,2023

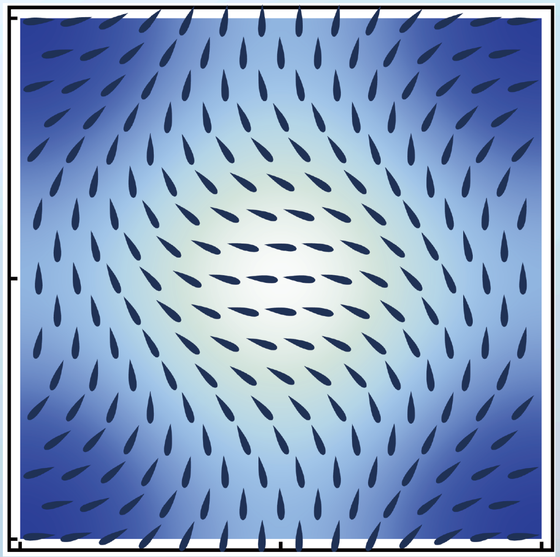
$$H_0 = t \sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} \begin{pmatrix} -\mu & -ie^{i\alpha_k^\uparrow} & 0 & 0 \\ ie^{-i\alpha_k^\uparrow} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha_k^\downarrow} \\ 0 & 0 & -ie^{i\alpha_k^\downarrow} & -\mu \end{pmatrix} c_{\boldsymbol{k}}$$

$$\alpha_k^\sigma = \eta_\sigma \left(\cos k_x + \cos k_y\right)$$





$b(\boldsymbol{k})$



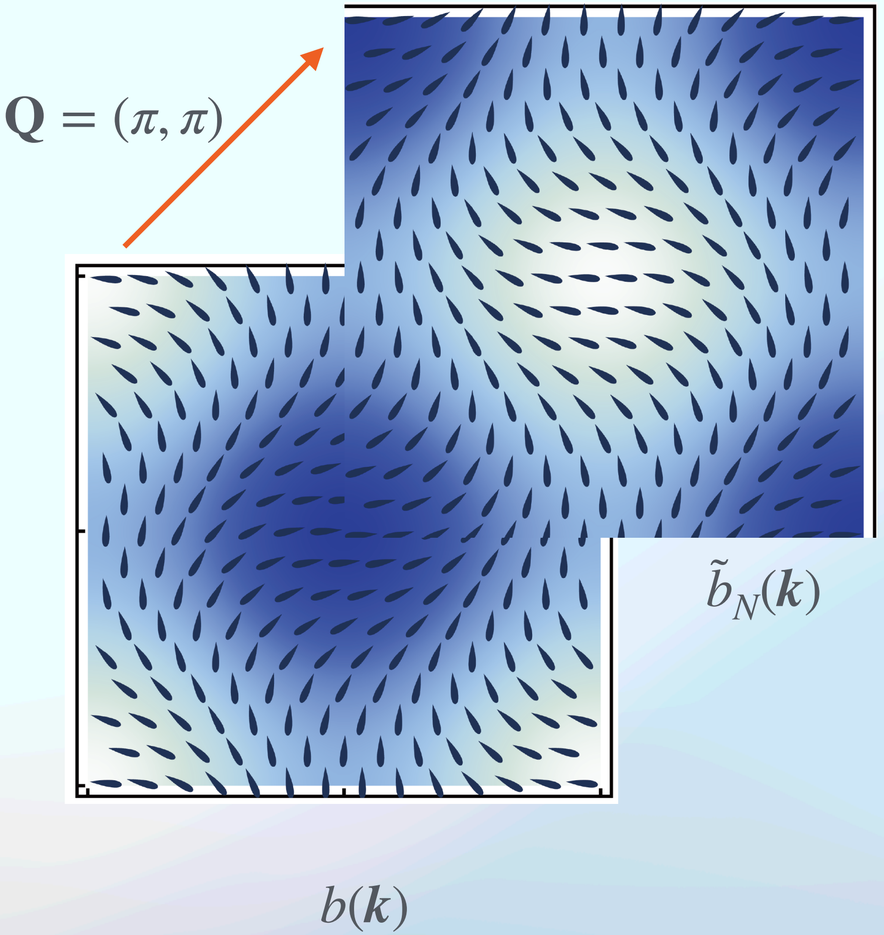
$\tilde{b}_N(\boldsymbol{k})$

Example

JS Hofmann *et al*, 2022,2023

$$H_0 = t \sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} \begin{pmatrix} -\mu & -ie^{i\alpha_k^\uparrow} & 0 & 0 \\ ie^{-i\alpha_k^\uparrow} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha_k^\downarrow} \\ 0 & 0 & -ie^{i\alpha_k^\downarrow} & -\mu \end{pmatrix} c_{\boldsymbol{k}}$$

$$\alpha_k^\sigma = \eta_\sigma \left(\cos k_x + \cos k_y\right)$$



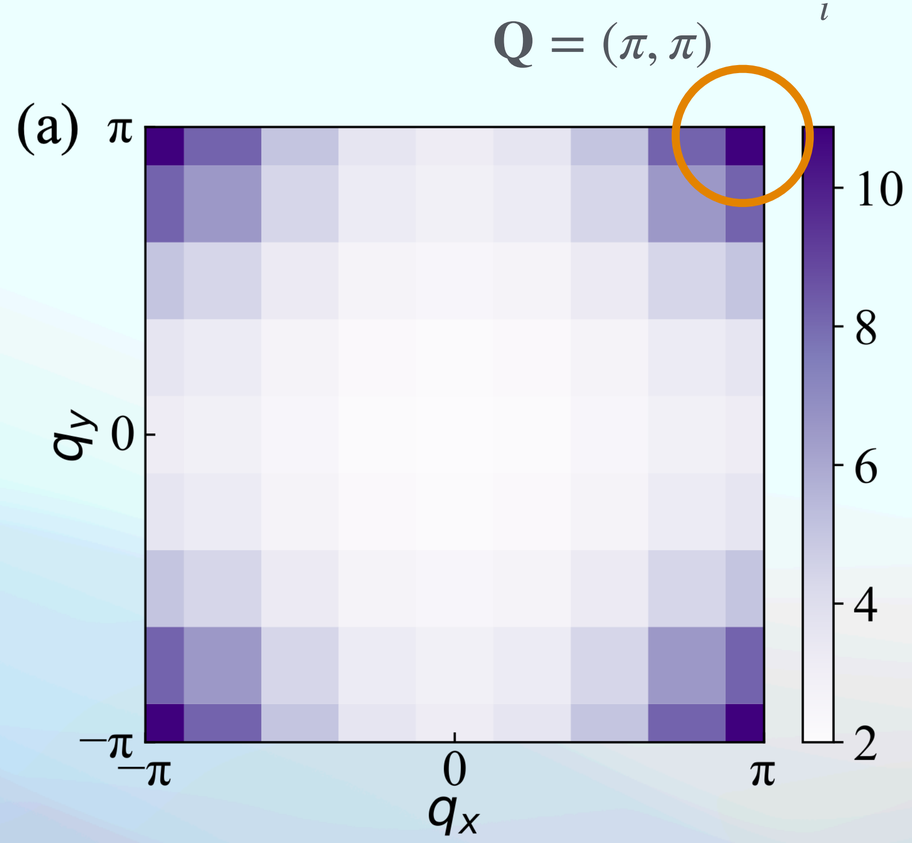
QGN!

QMC Simulation

Include Hubbard interaction $H = H_0 + U \sum_i \left(n_{i,A}^2 + n_{i,B}^2 \right)$

Use dQMC for 8x8 lattice

U/t=1



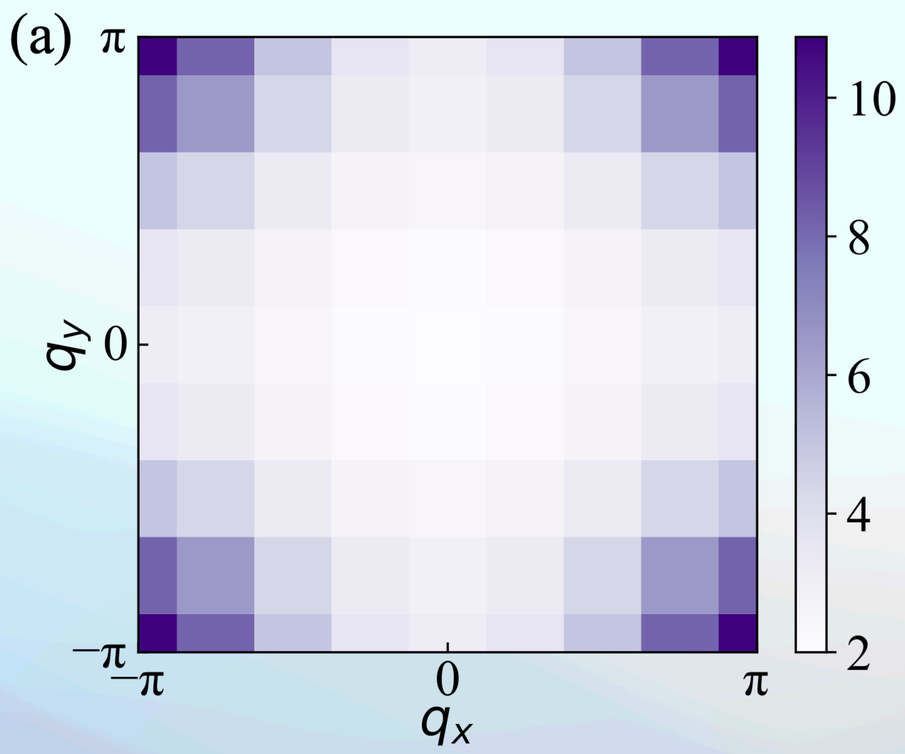
T/t = 0.05

QMC Simulation

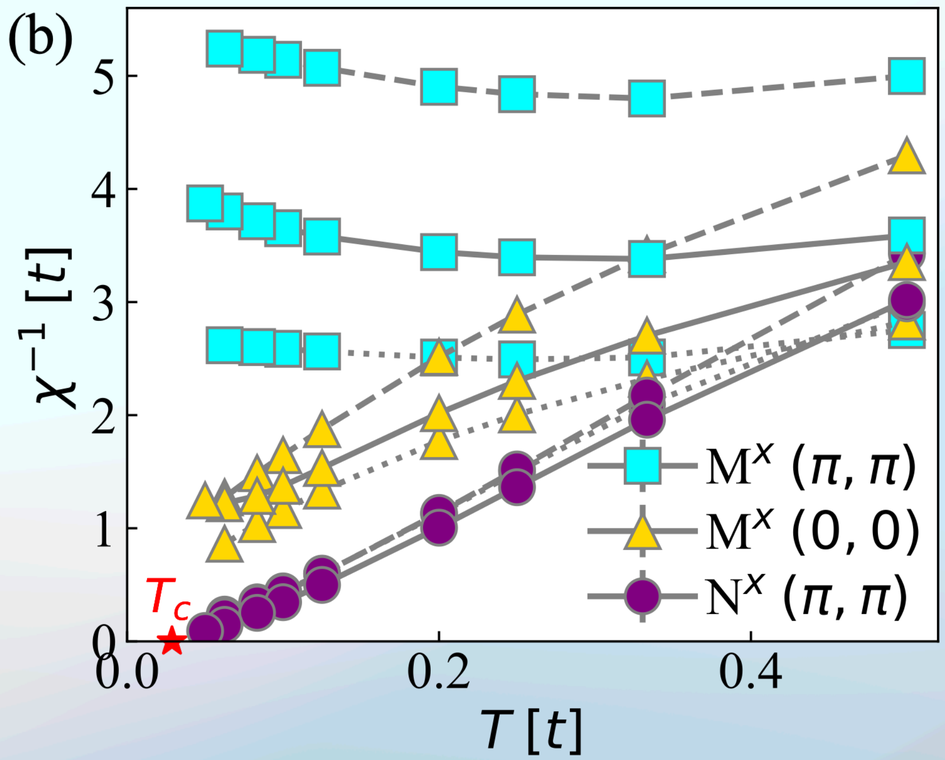
Include Hubbard interaction $H = H_0 + U \sum_i \left(n_{i,A}^2 + n_{i,B}^2 \right)$

Use dQMC for 8x8 lattice

U/t=1



T/t = 0.05



Superconductivity

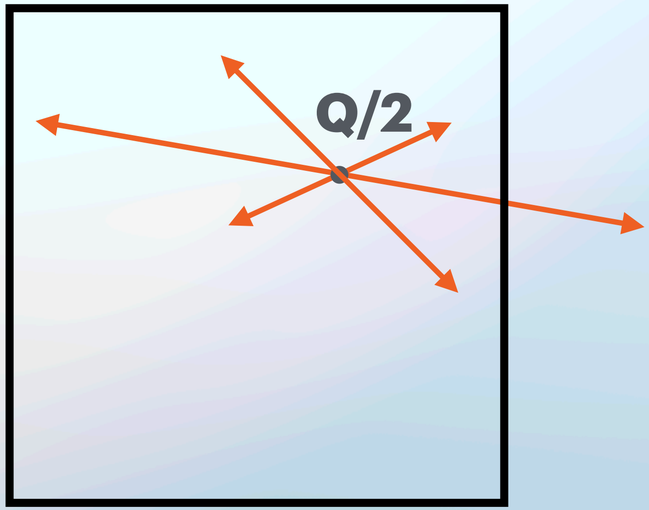
With attractive interactions in the flat band, can get exotic superconductivity

$$\chi_{\boldsymbol{Q}}^{\mathrm{pp}} = \frac{1}{4T}\frac{|\nu|}{\operatorname{arctanh}|\nu|}\sum_{\boldsymbol{k}} o_0^2(\boldsymbol{k})\left(Nn^2 + \frac{1}{2}\boldsymbol{b}^R(\boldsymbol{k}+\boldsymbol{Q})\cdot\boldsymbol{b}(-\boldsymbol{k})\right)$$

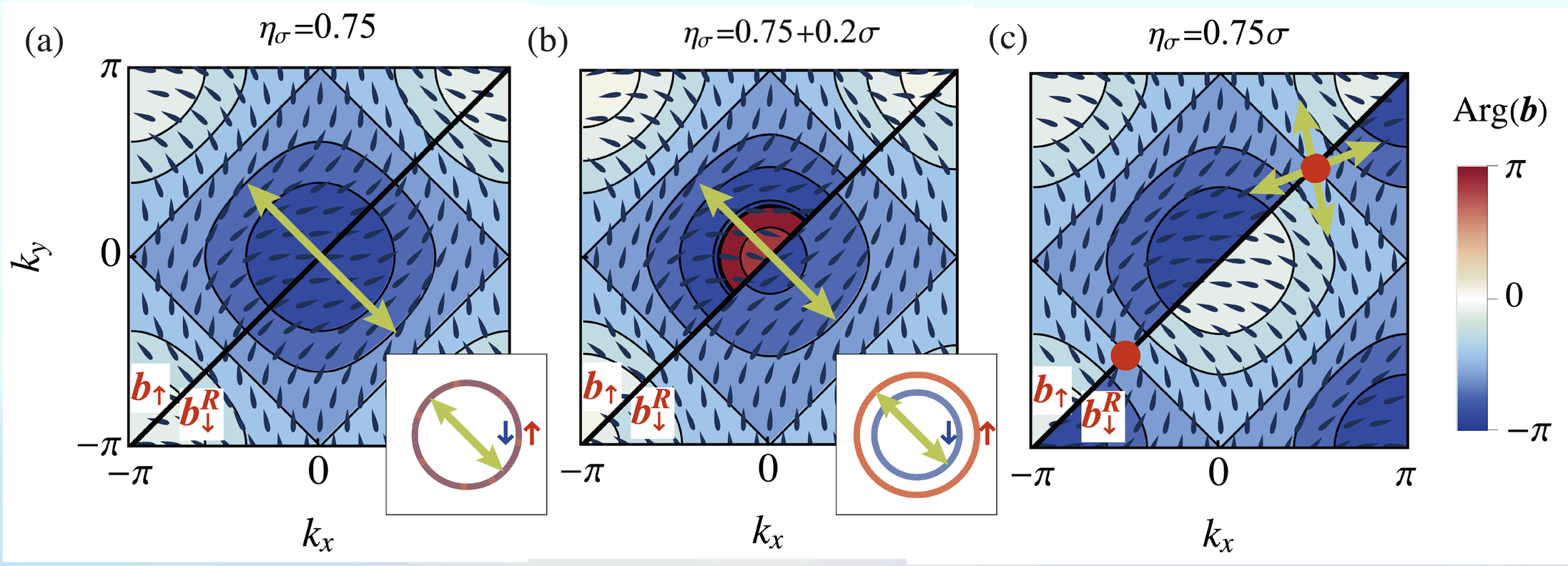
Similar nesting conditions

$$\boldsymbol{b}^R(\boldsymbol{k}+\boldsymbol{Q}) \parallel \boldsymbol{b}(-\boldsymbol{k})$$

$$\boldsymbol{b}^R\left(\frac{\boldsymbol{Q}}{2}+\frac{\boldsymbol{k}}{2}\right) \parallel \boldsymbol{b}\left(\frac{\boldsymbol{Q}}{2}-\frac{\boldsymbol{k}}{2}\right)$$



Superconducting QGN

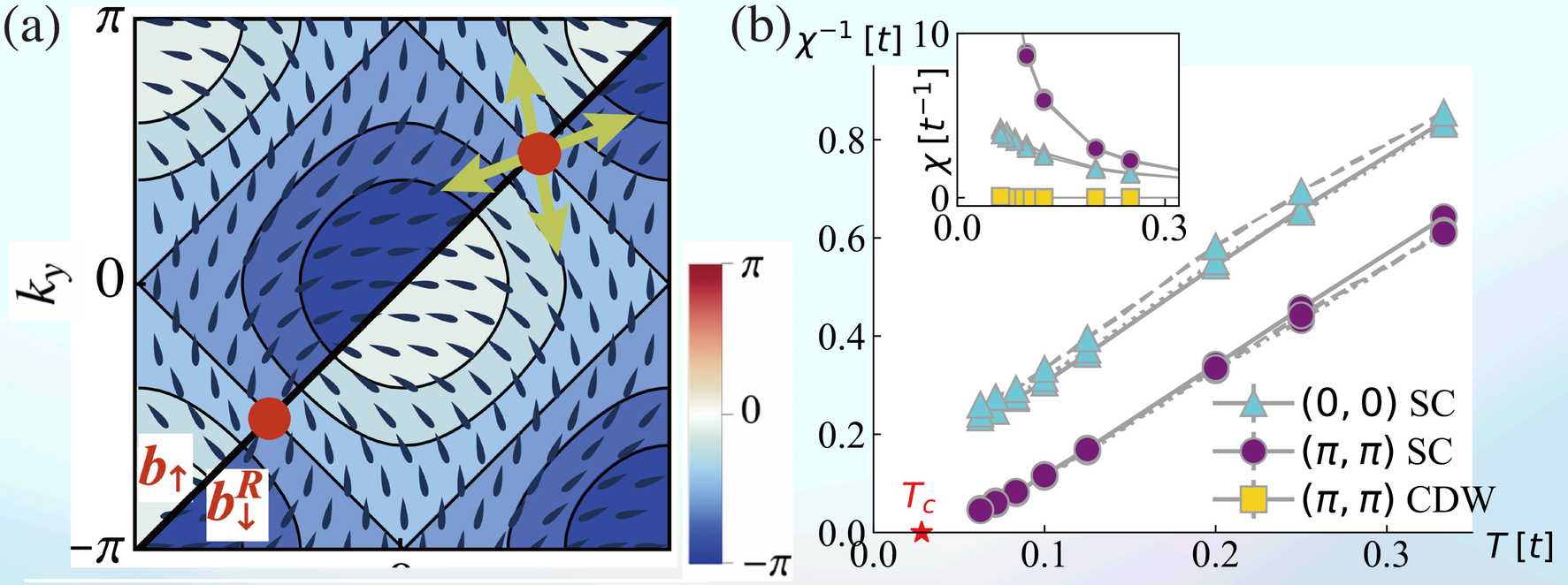


$\boldsymbol{Q} = \mathbf{0}$

$\boldsymbol{Q} \sim \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\boldsymbol{Q} = (\pi, \pi)$

QMC

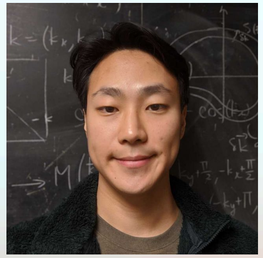


$$\eta_\sigma = 0.75\sigma$$

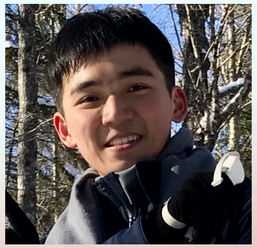
Summary

- We developed an algebraic approach to quantum geometric nesting
- It allows intuitive study of ordering instabilities of flat bands
- QMC validates the approach for a simple model system
- We also obtain relations between a “high temperature stiffness” and a generalized quantum metric (for both particle/hole and superconducting orders at any $\mathbf{Q}$)
- ?? Is there a way to include *non-symmetry breaking “orders”* like FQAHE?

Kinetic theory

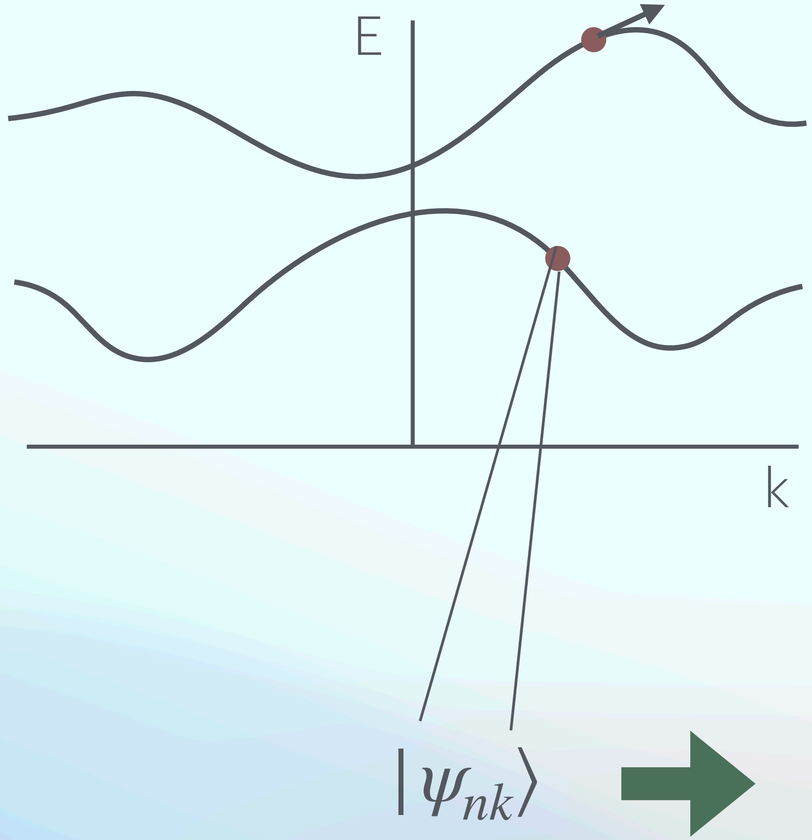


Takamori Park
UCSB -> Princeton



Xiaoyang Huang
CU Boulder -> PI

Bands



$\epsilon_n(k)$ → $\mathbf{v}_n(k)$ → $\sigma_{xx} \sim \langle v_n^x v_n^x \rangle$

Energetics

Geometry

$\Omega_{n,\alpha\beta}(k)$ Berry curvature → $\sigma_{xy} \sim \langle \Omega \rangle$

$g_{n,\alpha\beta}(k)$ Quantum metric

Souza, Wilkens, Martin 2000
Various suggestions

Role in interacting systems?

Linear response

A host of “conventional” (but still complex) transport coefficients

$$\mathbf{j} = \mathbf{L}^{11}\mathcal{E} + \mathbf{L}^{12}(-\nabla T),$$
$$\mathbf{j}^q = \mathbf{L}^{21}\mathcal{E} + \mathbf{L}^{22}(-\nabla T),$$

Ashcroft+Mermin

And some originating from Berry curvature

Anomalous Hall effect

$$\boldsymbol{j}_e^{ah} = e^2 \sum_n \int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\, n_F(\omega_n(\boldsymbol{k}))\boldsymbol{\Omega}_n(\boldsymbol{k}) \times \boldsymbol{E}$$

Karplus+Luttinger, 1954!

Gyromagnetic effect

$$\alpha_{ij}^{\mathrm{GME}} = \frac{i\omega\tau e}{1-i\omega\tau} \sum_n \int [d\boldsymbol{k}](\partial f/\partial\varepsilon_{\boldsymbol{k}n}) v_{\boldsymbol{k}n,i} m_{\boldsymbol{k}n,j}$$

Ma+Pesin 2015; Song *et al* 2016

Chiral anomaly…

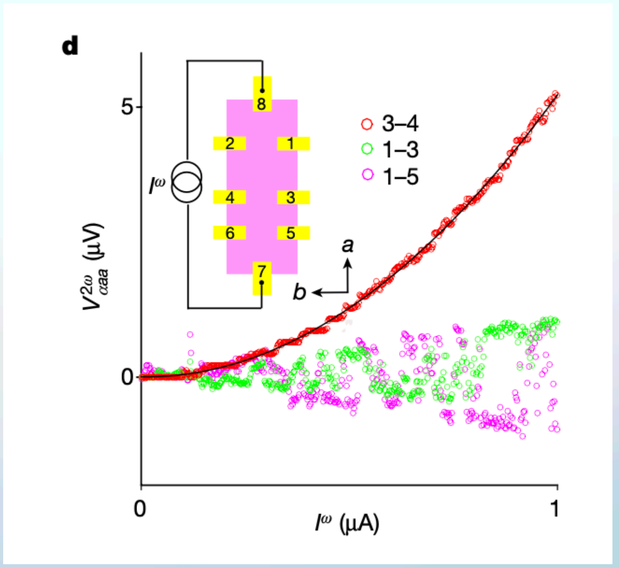
Non-linear response

Non-linear Hall effect

$$j_a^0 = \chi_{\mathrm{abc}} \mathcal{E}_b \mathcal{E}_c^*,\ j_a^{2\omega} = \chi_{\mathrm{abc}} \mathcal{E}_b \mathcal{E}_c,$$

$$\chi_{\mathrm{abc}} = -\varepsilon_{adc} \frac{e^3 \tau}{2(1 + i\omega\tau)} \int_k f_0 (\partial_b \Omega_d)$$

Sodemann+Fu, 2015



Q. Ma *et al*, 2019 (WTe2)

Non-linear response

Quantum metric effects

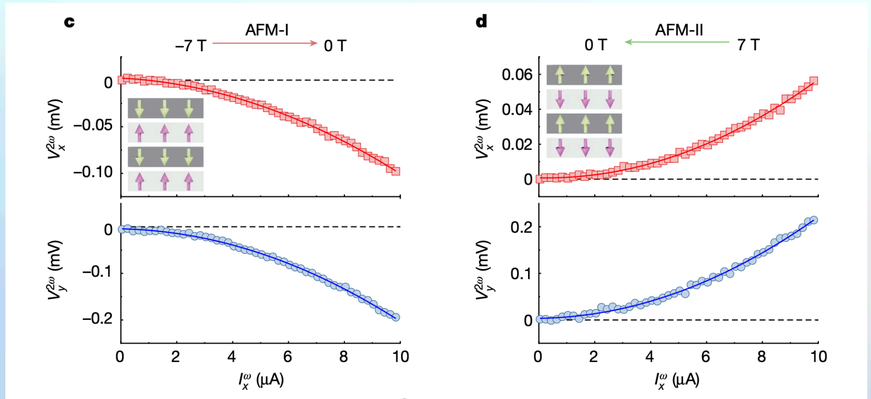
$$\sigma_{a;bc}^{\mathrm{BCPD}} = \frac{e^3}{\hbar} \sum_{m,p,\boldsymbol{k}} \int [d\boldsymbol{k}] f_m \left[\partial_a \tilde{\mathcal{G}}_{mp}^{bc} + \partial_b \tilde{\mathcal{G}}_{mp}^{ac} + \partial_c \tilde{\mathcal{G}}_{mp}^{ab} \right].$$

"Band-resolved quantum metric dipole"

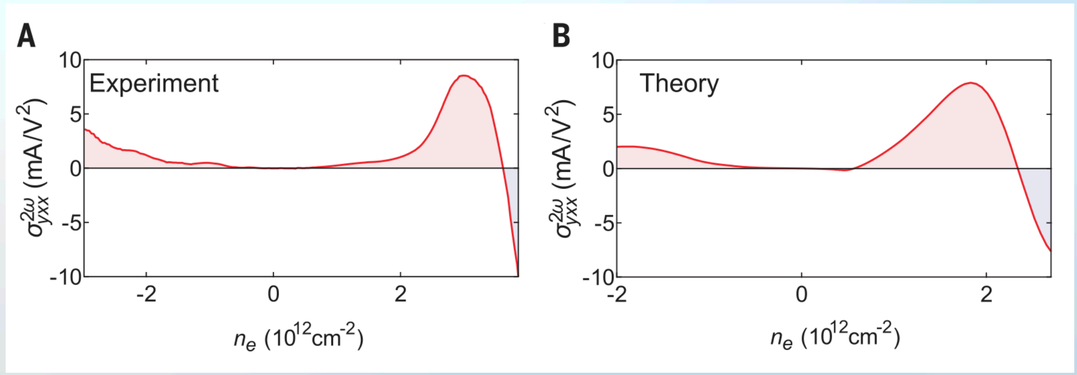
K. Das *et al*,2023

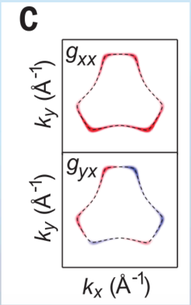
D. Kaplan *et al*, 2024

$MnBi_2Te_4$



N. Wang *et al*, 2023





A. Gao *et al*, 2023

Quadrupole response

$$j^{\mu}_{\text{geom.}}(\mathbf{r}) = -\frac{Q^2}{2\hbar}\int \frac{d^D\mathbf{K}}{(2\pi)^D}\, f_0(\mathbf{K}) E^{(0)}_{\nu\lambda} \frac{\partial g^{\nu\lambda}(\mathbf{K})}{\partial K_\mu}$$

Quantum metric dipole

M. Lapa + T. Hughes, 2019

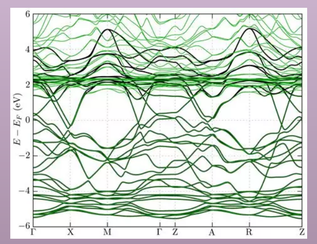
$$E_{\mu\nu} = \partial E_\mu / \partial x_\nu$$

Semiclassical approach

Two approximations, done separately

c.f. Great talk from Jenny Coulter

Band structure



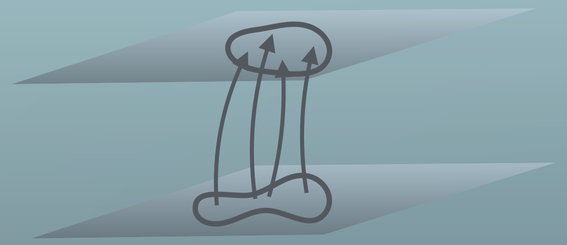
$$|\psi_{nk}\rangle$$

Single particle equations of motion

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}_n = \boldsymbol{\nabla}_{\boldsymbol{k}}\tilde{\epsilon}_{n\boldsymbol{k}} - \frac{d\boldsymbol{k}}{dt} \times \boldsymbol{\Omega}_{n\boldsymbol{k}},$$
$$\frac{d\boldsymbol{k}}{dt} = \boldsymbol{F}_n = -e\boldsymbol{E} - e\frac{d\boldsymbol{x}}{dt} \times \boldsymbol{B}.$$

$$[x, k] = i\hbar$$

Boltzmann equation



$$\partial_t f_n + \frac{dx_\mu}{dt}\frac{\partial f_n}{\partial x_\mu} + \frac{dk_\mu}{dt}\frac{\partial f_n}{\partial k_\mu} = \mathcal{C}[f]$$

$$\begin{pmatrix} \rho_1 & \rho_{12} \\ \rho_{21} & \rho_2 \end{pmatrix} \rightarrow \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$$

What we did

Systematic exact (order by order in spatial gradients) derivation of kinetics from the density matrix alone.

Just a hint of the (rather technical) method:

$$i\hbar\partial_t\rho = [H, \rho]$$

Matrix in *both* band and position/momentum space

$$\rho_{n,k;n',k'}$$

Wigner transform

$$i\hbar\partial_t\mathbf{F} = [\mathbf{H}, \mathbf{F}]_\star$$

$$F_{n,n'}(k, x)$$

Moyal diagonalization

QKE

$$f_n(k, x)$$

Kinetic equation

$$\partial_t f_n + \partial_\alpha \mathcal{J}_\alpha = 0$$

Just continuity equation for density in a band

c.f. simple convection $\mathcal{J}_\alpha = f_n v_\alpha$ → Boltzmann equation

Incompressible flow

Here, to second order:

$$\begin{aligned}\mathcal{J}_\alpha =& \varepsilon_{\alpha\beta} \operatorname{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu} h \partial_\beta g_{\nu\lambda}\right)\right] \\ &+ \hbar^2 \varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma \operatorname{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta} h g_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu} \operatorname{tr}\left[f \partial^3_{\beta\lambda\nu} h\right]\end{aligned}$$

Kinetic equation

$$\partial_t f_n + \partial_\alpha \mathcal{J}_\alpha = 0$$

Just continuity equation for density in a band

c.f. simple convection $\mathcal{J}_\alpha = f_n v_\alpha$ → Boltzmann equation

Incompressible flow

Here, to second order:

$$\begin{aligned}\mathcal{J}_\alpha =& \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu} h \partial_\beta g_{\nu\lambda}\right)\right] \\ &+ \hbar^2\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta} h g_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu} h\right]\end{aligned}$$

quantum metric $g_{\alpha\beta} = \frac{1}{2}\mathrm{diag}\left(\Lambda_\alpha\Lambda_\beta + \Lambda_\beta\Lambda_\alpha\right) - A_\alpha A_\beta = \frac{1}{2}\,\mathrm{tr}(\partial_\alpha P_n \partial_\beta P_n)$

Kinetic equation

$$\mathcal{J}_\alpha = \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h\Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1-\frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}h\partial_\beta g_{\nu\lambda}\right)\right]$$
$$+\hbar^2\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h\partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta}hg_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu}h\right]$$

- General and exact up to second order in semiclassical expansion
- Explicitly contains real space and momentum space, and capable of describing inhomogeneous systems. Magnetic field can be included by modifying $\epsilon_{\alpha\beta}$
- Quantum geometry of bands appears explicitly, takes local band Hamiltonian as input
- **Only band-intrinsic quantities appear**: but these are bands renormalized by quantum corrections.
- Ready to attack all sorts of problems!

Example: separable problem

$$\mathsf{H}(x,p) = \mathsf{H}_0(p) - eV(x)\mathsf{I}_N$$

Band form Diagonal potential

Kinetic equation (relaxation time approx)

$$\begin{aligned}\partial_t f =& -\partial_{x_i}\left[f\left(v_i + \frac{e}{\hbar}\Omega_{ij}\partial_{x_j}V + e\mathsf{T}_{ij}\partial^2_{tx_j}V + \frac{e^2}{\hbar}\left(\partial_{k_j}\mathsf{T}_{il} - \frac{1}{2}\partial_{k_i}\mathsf{T}_{jl}\right)\partial_{x_j}V\partial_{x_l}V + \frac{e}{\hbar}\left(\hbar v_j\mathsf{T}_{il} - \frac{1}{2}\partial_{k_i}\mathsf{g}_{jl}\right)\partial^2_{x_jx_l}V\right)\right] \\ &- \frac{e}{\hbar}\partial_{k_i}f\partial_{x_i}V + \frac{e}{2\hbar}\partial^2_{x_ix_l}(f\partial_{k_j}\mathsf{g}_{il}\partial_{x_j}V) - \frac{e}{\hbar}\partial^2_{k_ix_j}(f\mathsf{g}_{jl}\partial^2_{x_ix_l}V) + \frac{1}{24\hbar}\partial^3_{x_ix_jx_l}f\partial^3_{k_ik_jk_l}\varepsilon \\ &+ \frac{e}{24\hbar}\partial^3_{k_ik_jk_l}f\partial^3_{x_ix_jx_l}V + \mathcal{I}[f] + \mathcal{O}(\partial_x^4). \end{aligned} \tag{3.29}$$

$$T_{n;\mu\nu} = -\sum_{m\neq n}\frac{\langle\partial_{p_\mu}u_n|u_m\rangle\langle u_m|\partial_{p_\mu}u_n\rangle + (\mu\leftrightarrow\nu)}{\epsilon_n - \epsilon_m}$$

"band-normalized quantum metric"

(*not* a purely geometric quantity)

Results

- Reproduced
  - Non-linear Hall effect
  - Non-linear band-resolved quantum metric dipole contributions
  - Quadrupole response
- New results in collision less regime
  - e.g. polarization function

} General form agrees with literature but we found errors

$$\widehat{C}^{(2)}(\omega,q) = -\mathrm{i}e^{-1}\hbar\int_k \frac{\delta\widehat{f}_{e=1}(q,k,\omega)}{\delta\widehat{V}(q,\omega)} \approx -\mathrm{i}\int_k \left[\frac{\hbar^2 n_F' \vec{v}\cdot\vec{q}\,(1-\mathrm{g}_{ij}(k)q_iq_j)}{\hbar\omega-\varepsilon(k+q/2)+\varepsilon(k-q/2)} + \hbar n_F \mathrm{t}_{ij}(k)q_iq_j\right]$$

Summary

$$\mathcal{J}_\alpha = \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h\Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}h\partial_\beta g_{\nu\lambda}\right)\right]$$
$$+ \hbar^2\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h\partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta}hg_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu}h\right]$$

- We found a procedure to obtain semiclassical electron dynamics organized by spatial gradients
- We found a general result for the phase space current to *second* order in gradients, **capable of handling inhomogeneous systems and magnetic fields**
- Prior results for linear and non-linear response are recovered, and in some cases corrected
- New results for collision less regime
- Still needed: a scattering/interactions theory with the same validity

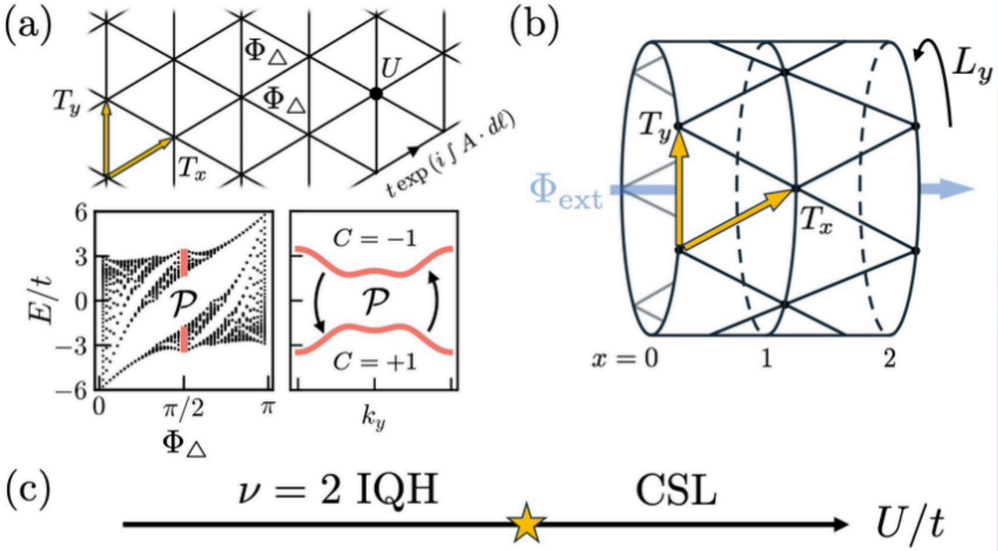
Brief update: anyon superconductivity?

Chiral Spin Liquid and Quantum Phase Transition in the Triangular Lattice Hofstadter-Hubbard Model

Stefan Divic,[1,2] Tomohiro Soejima,[3] Valentin Crépel,[2] Michael P. Zaletel,[1,4] and Andrew Millis[2,5]

Anyon Superconductivity from Topological Criticality in a Hofstadter-Hubbard Model

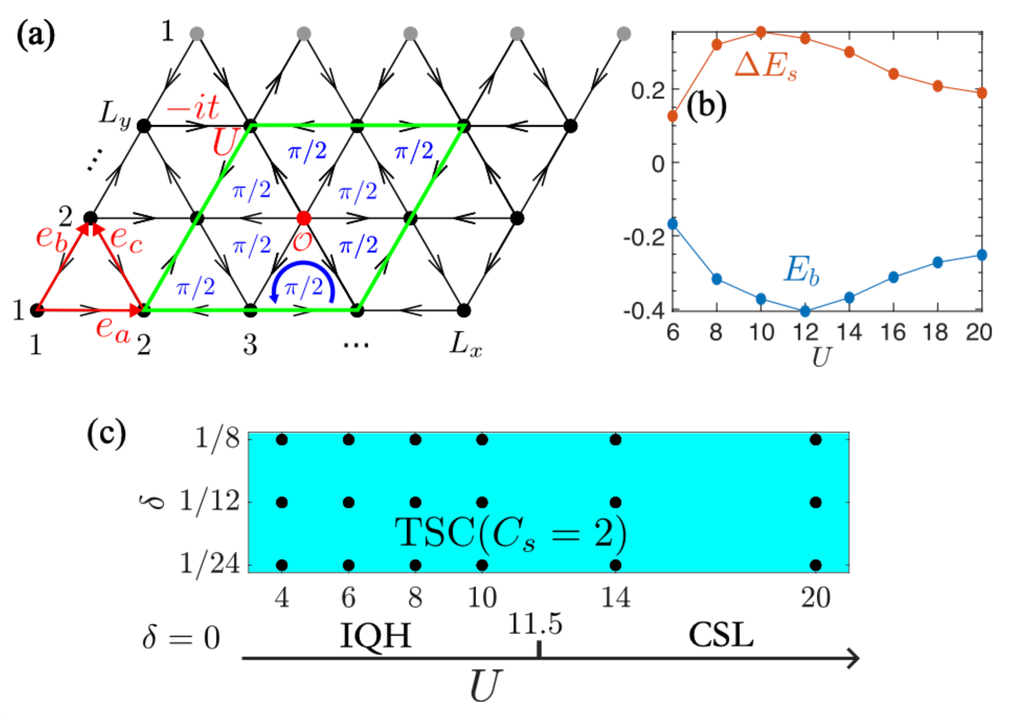
Stefan Divic,[1] Valentin Crépel,[2] Tomohiro Soejima (副島智大),[3] Xue-Yang Song,[4] Andrew J. Millis,[2,5] Michael P. Zaletel,[1,6] and Ashvin Vishwanath[3]

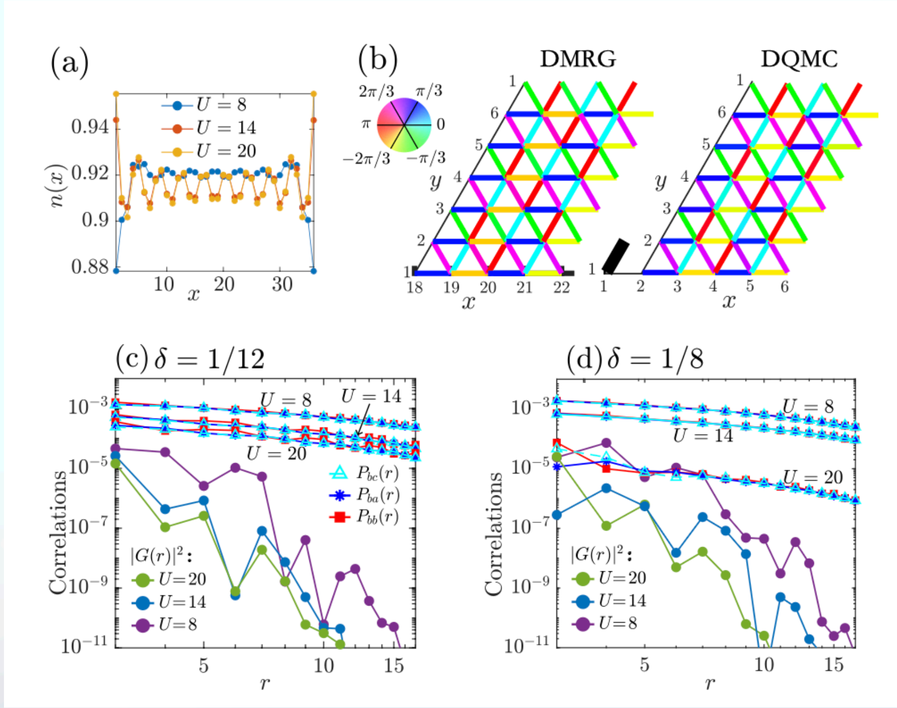


Charge gap must vanish at this QCP, but 1-e gap does not.

Strong indication that paired state arises on doping.

Our study





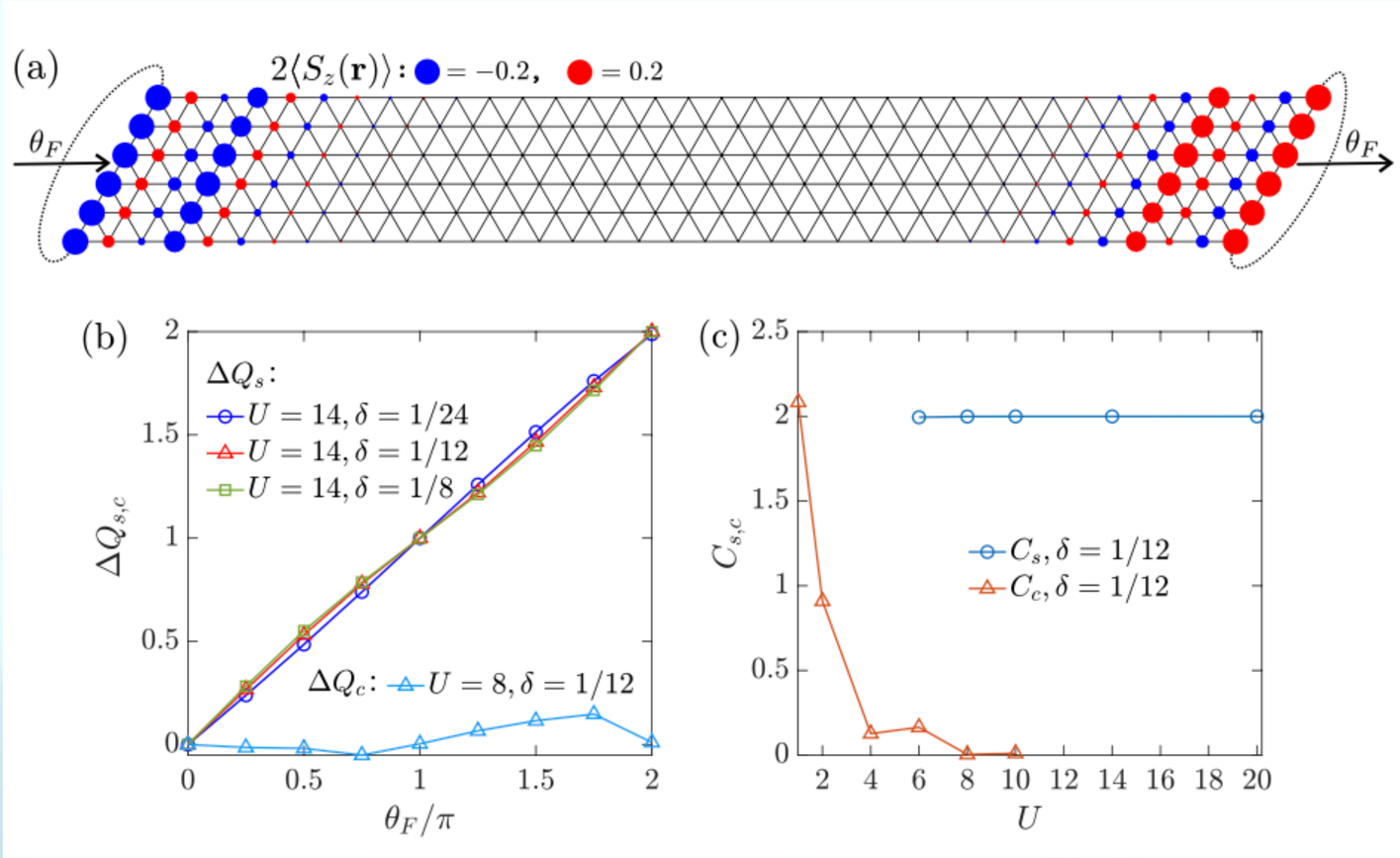
Strong pairing correlations

c.f.

Robust superconductivity upon doping chiral spin liquid and Chern insulators in a Hubbard-Hofstadter model

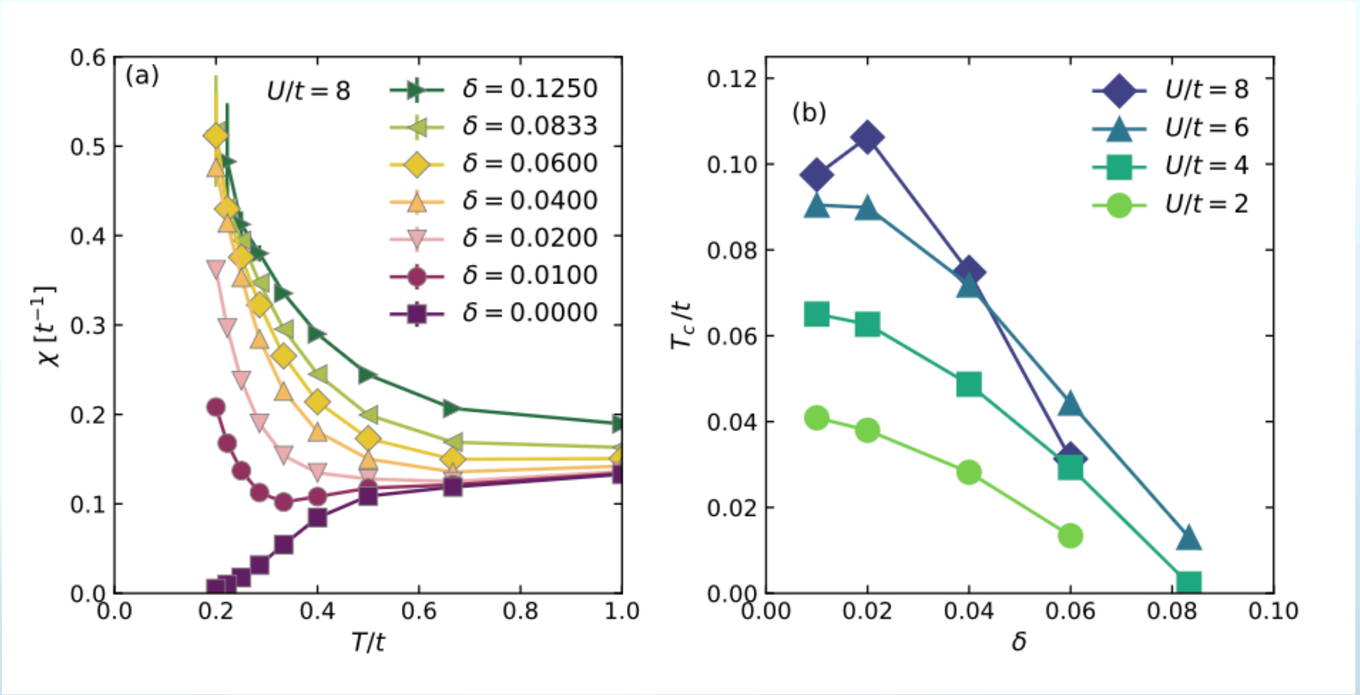
Clemens Kuhlenkamp,[1,*] Stefan Divic,[2,3,*] Michael P. Zaletel,[3,4]
Tomohiro Soejima (副島智大),[1] and Ashvin Vishwanath[1]

Topology: spin pumping



DQMC

Sign problem, but manageable

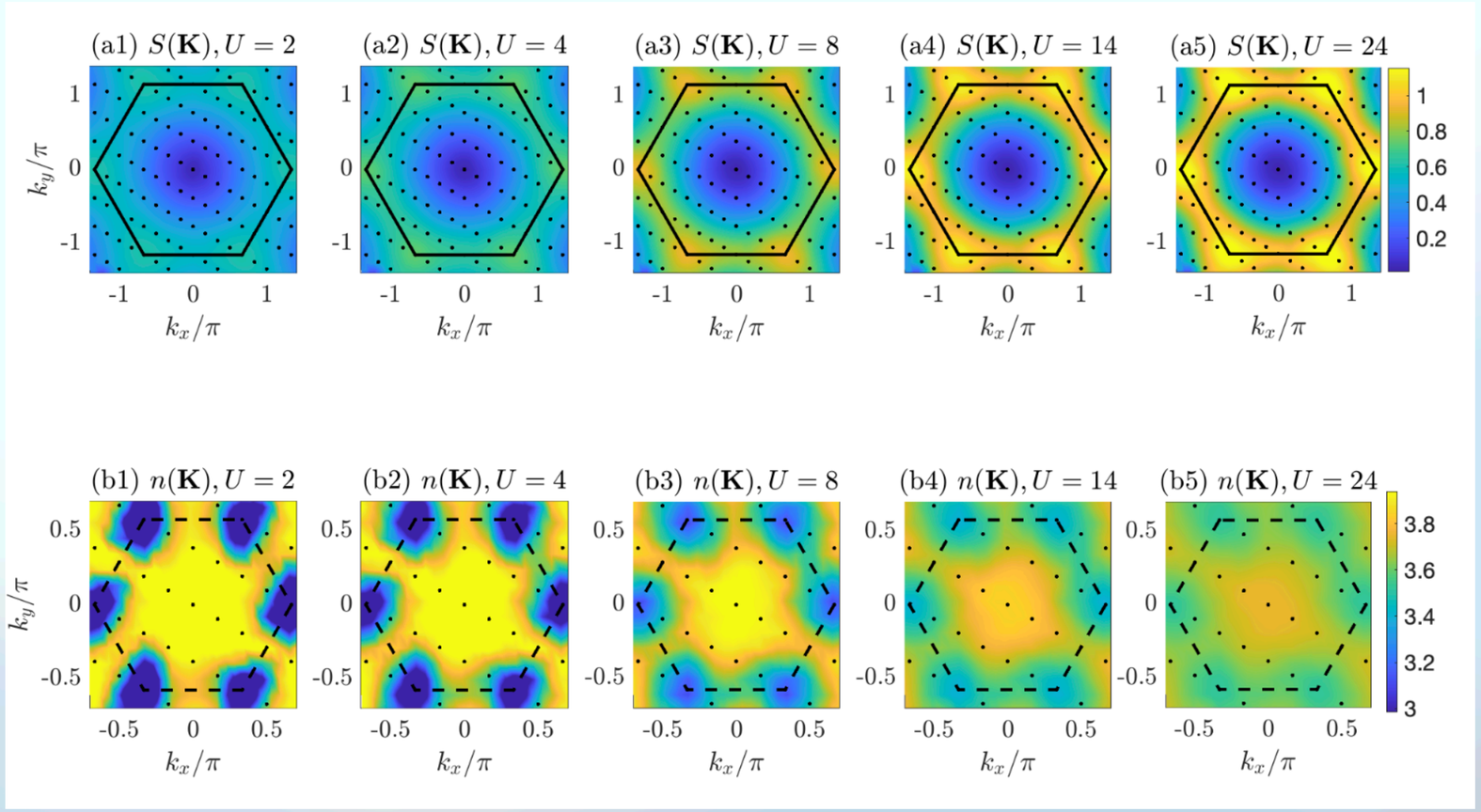


BCS or not?

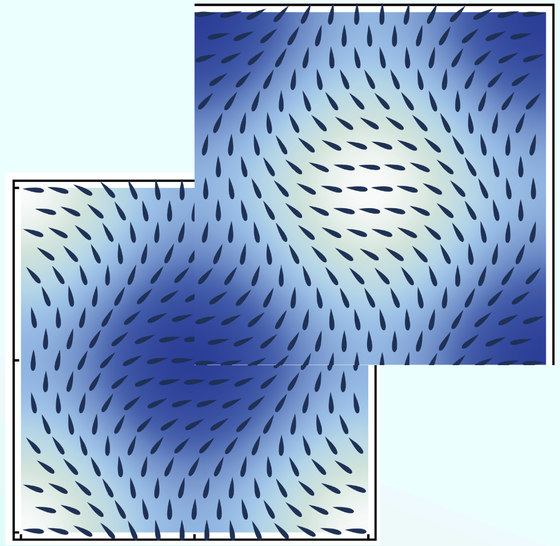
All in the superconducting phase, but we can see the evolution

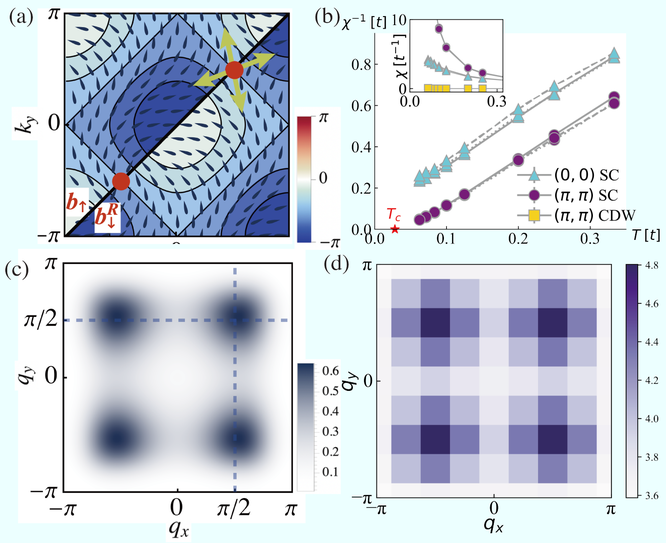
Spin S(k)

Occupation number n(k)

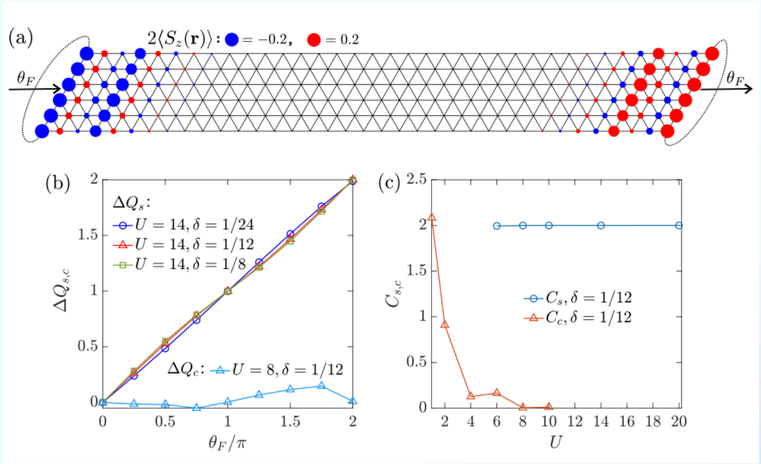


Thanks





arXiv: 2504.03882



arXiv: 2509.02757

$$\mathcal{J}_\alpha = \varepsilon_{\alpha\beta}\,\mathrm{tr}\left[f\left(\partial_\beta h + \hbar\left(\varepsilon_{\sigma\lambda}\partial_\sigma h\Omega_{\lambda\beta} - \Omega_{t\beta}\right)\left(1 - \frac{\hbar}{2}\Omega_{\mu\nu}\right) + \frac{\hbar^2}{2}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}h\partial_\beta g_{\nu\lambda}\right)\right]$$
$$+ \hbar^2\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\partial_\sigma\,\mathrm{tr}\left[f\left(\frac{1}{2}\varepsilon_{\mu\nu}\partial_\mu h\partial_\nu g_{\beta\lambda} - \frac{1}{2}\partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu}\partial^2_{\mu\beta}h g_{\nu\lambda}\right)\right] - \frac{\hbar^2}{24}\varepsilon_{\alpha\beta}\varepsilon_{\sigma\lambda}\varepsilon_{\mu\nu}\partial^2_{\sigma\mu}\,\mathrm{tr}\left[f\partial^3_{\beta\lambda\nu}h\right]$$

arXiv: 2504.10447

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH · UNITED STATES AIR FORCE

GORDON AND BETTY MOORE FOUNDATION

NSF

CIFAR CANADIAN INSTITUTE FOR ADVANCED RESEARCH · ICRA INSTITUT CANADIEN DE RECHERCHES AVANCÉES

Simons Collaboration on Ultra-Quantum Matter