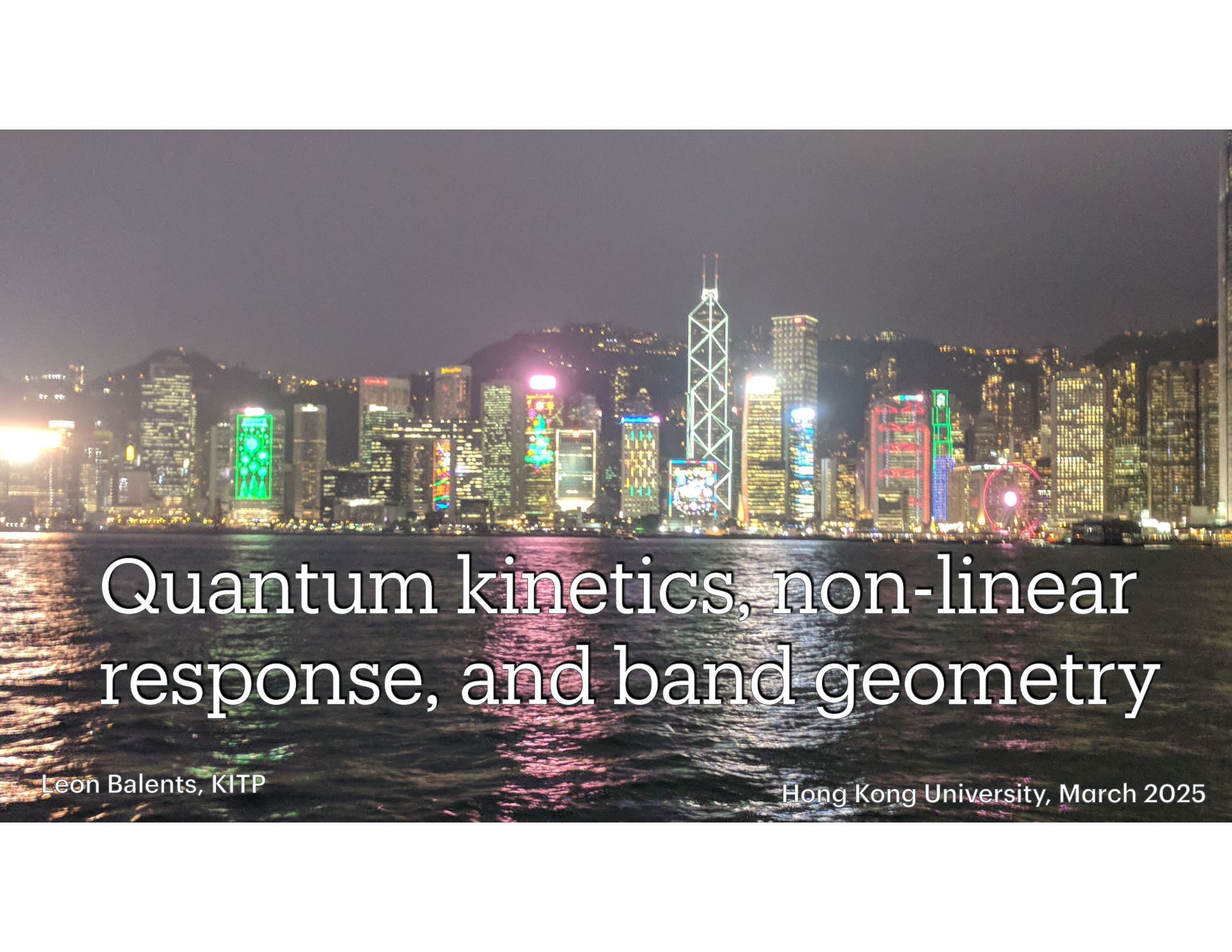




Quantum kinetics, non-linear response, and band geometry

Leon Balents, KITP

Hong Kong University, March 2025

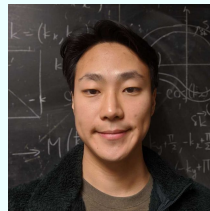
Collaborators



Lucile Savary
FACTS



Léo Mangeolle
TUM



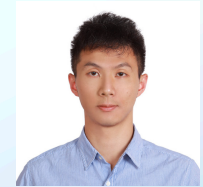
Takamori Park
UCSB



Xiaoyang Huang
CU Boulder



Kaixiang Su
UCSB



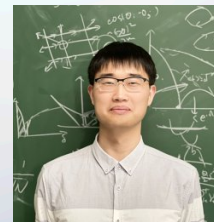
Ren-Bo Wang
FACTS



Prequel: PRB 2024



Wen Wang
UCSB

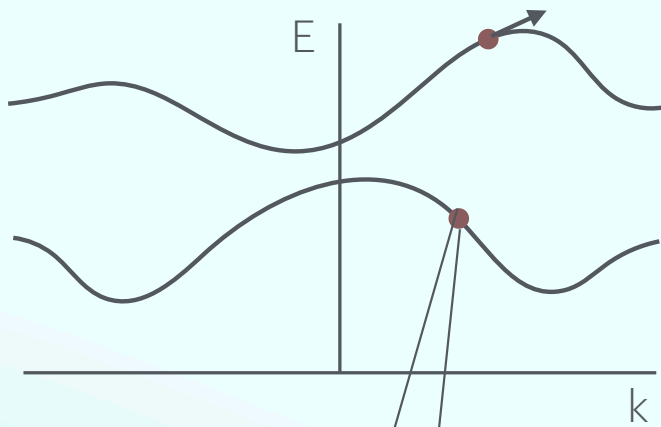


Jia-Xin Zhang
FACTS

Outline

- Bands and quantum geometry
- Quantum geometric nesting in flat bands
- Doing semiclassics correctly in dispersive bands with quantum geometry

Bands



$|\psi_{nk}\rangle$

➔
Geometry

$\epsilon_n(k)$ ➔ $\mathbf{v}_n(k)$

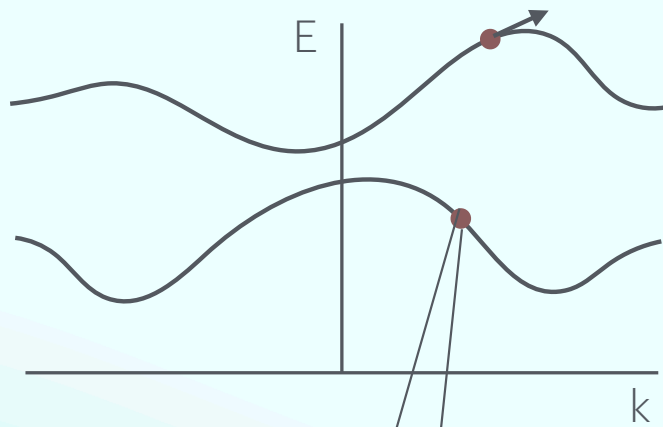
Energetics

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Quantum geometric nesting

Bands



$|\psi_{nk}\rangle$

Geometry

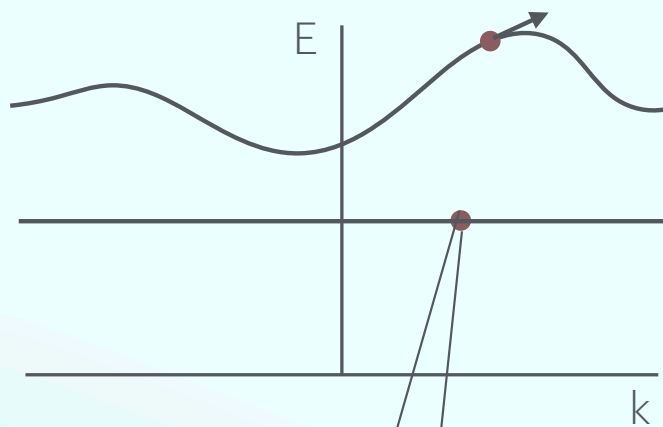
$$\epsilon_n(k) \rightarrow \mathbf{v}_n(k)$$

Energetics

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Flat bands



$|\psi_{nk}\rangle$

Geometry

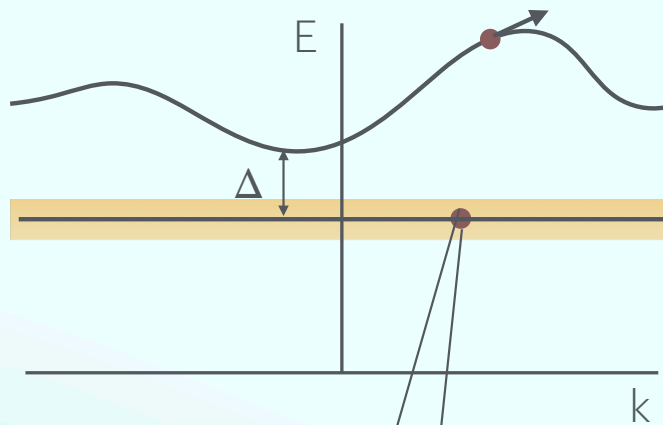


In the flat band limit, there is *only* geometry

$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

Flat bands



In the flat band limit, there is *only* geometry

Interaction $U \ll \Delta$

How does geometry determine the instabilities?

$|\psi_{nk}\rangle$

Geometry

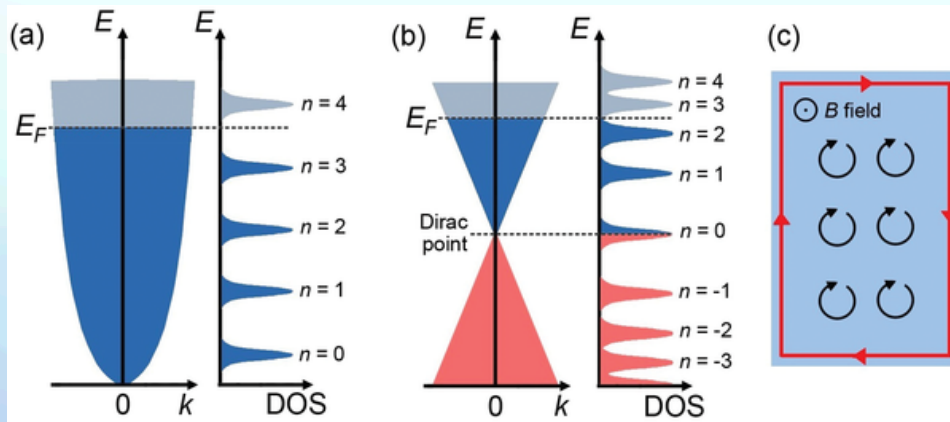
$\Omega_{n,\alpha\beta}(k)$ Berry curvature

$g_{n,\alpha\beta}(k)$ Quantum metric

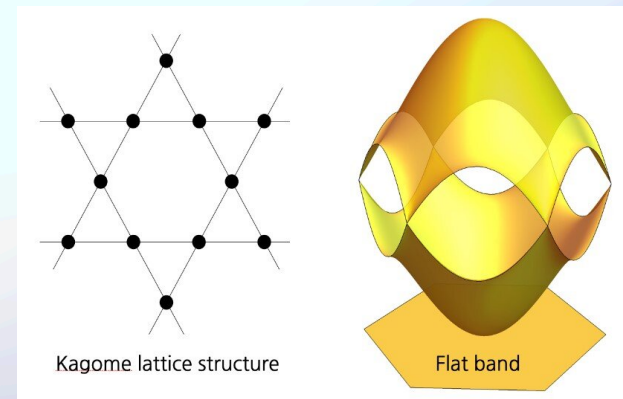
Flat bands

Generally a result of interference in electron motion, which can be induced by frustration or by magnetic flux.

Landau levels

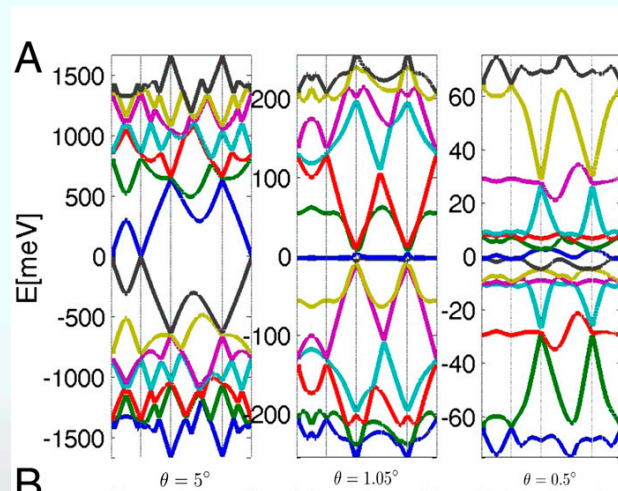
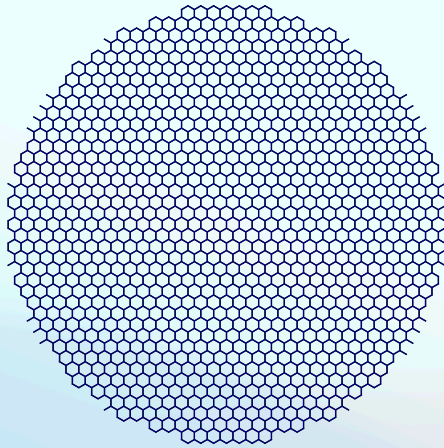


Frustrated hopping



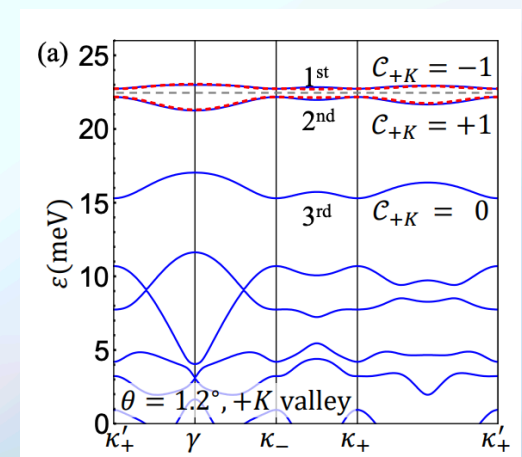
Flat bands

moiré



Bistritzer+Macdonald, 2011

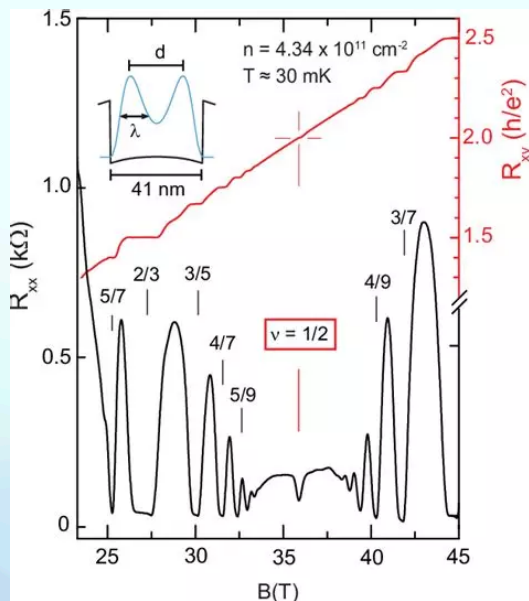
Graphene



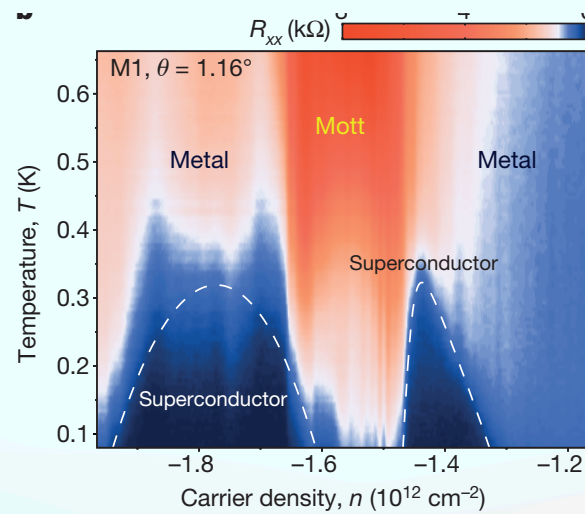
F. Wu *et al*, 2019

TMDs

Correlation physics

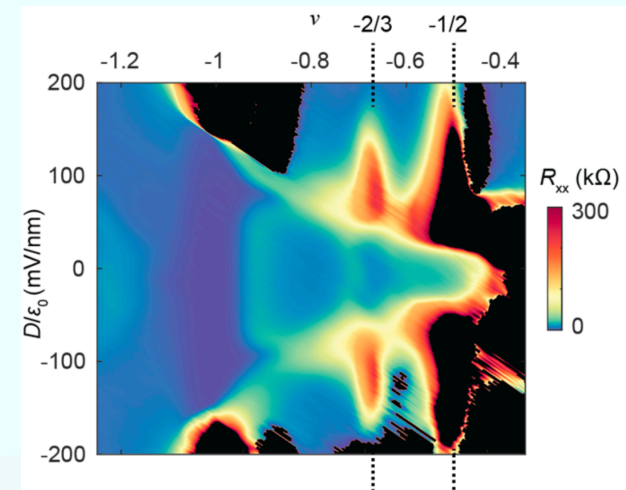


FQHE



Cao *et al*, 2018

Superconductivity and correlated insulators in TBG



H. Park *et al*, 2023

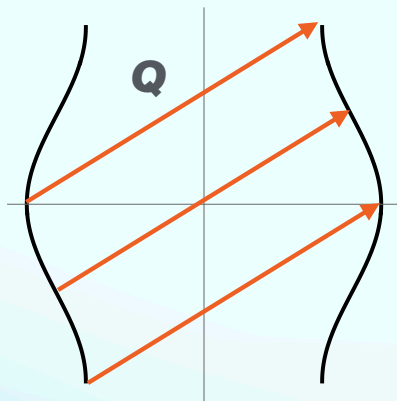
FQAHE in twisted MoTe2

So many different orders

- Ferromagnetism
- Antiferromagnetism
- Intervalley coherence
- Fractional quantum Hall effect
- Stripes
- Charge density wave
- Loop currents
- Superconductivity (s-wave, p-wave, d-wave, FFLO...)

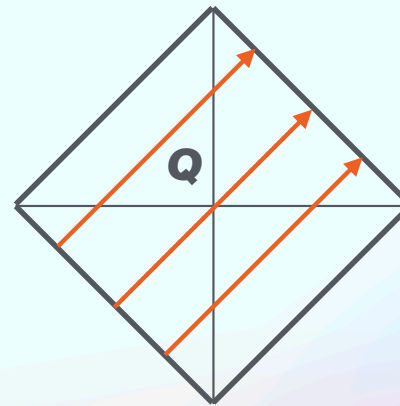
Nesting

Within a *dispersive* band, Fermi surface structure controls instabilities



Quasi-1d

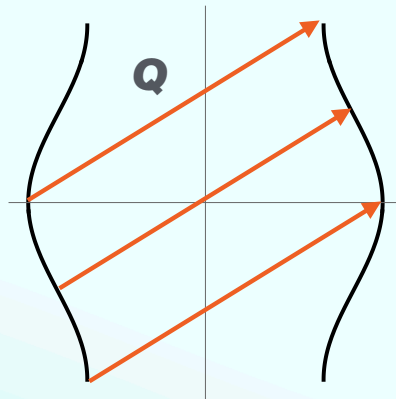
" $2k_F$ " CDW/SDW



Square lattice Hubbard

(π, π) antiferromagnet

Susceptibility



Restricts to near Fermi surface

$$\chi(Q, \omega) \sim \sum_k \frac{n_F(\epsilon_{k+Q}) - n_F(\epsilon_k)}{\omega - \epsilon_k + \epsilon_{k+Q}}$$

Enhanced susceptibility
when this vanishes for
extended range of k

What happens when the band is flat?? No Fermi surface!

Quantum geometric nesting

Z. Han et al, 2024

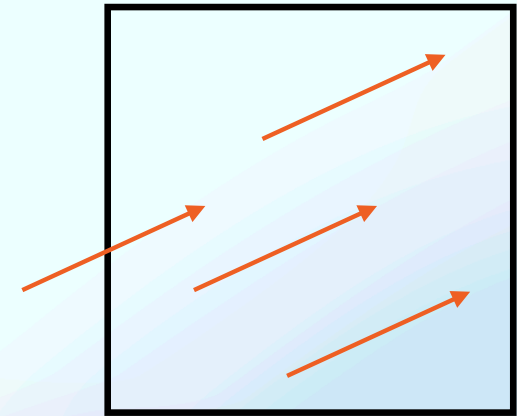
For flat band (or bands) at the Fermi energy, susceptibility is rendered finite only by thermal fluctuations. But there is still structure.

$$\chi_{\mathbf{Q}} = \frac{1}{T} \sum_{\mathbf{k}} \text{Tr} [\mathcal{O}^\dagger(\mathbf{k}) P(\mathbf{k} + \mathbf{Q}) \mathcal{O}(\mathbf{k}) P(\mathbf{k})] .$$

Order parameter

Projector

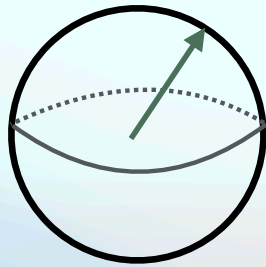
$$P(\mathbf{k}) = \sum_{n \in \text{FB}} |\psi_{n\mathbf{k}}\rangle \langle \psi_{n\mathbf{k}}|$$



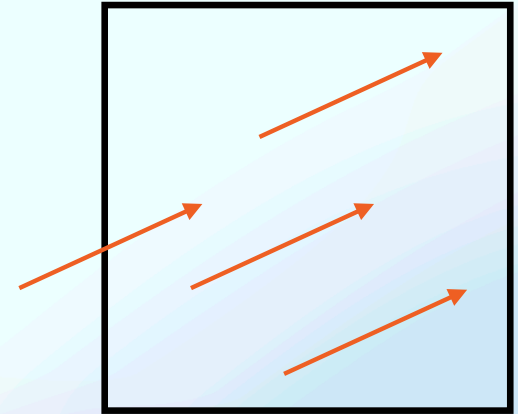
Quantum geometric nesting

Bloch vector representation

e.g. 2-band $P(\mathbf{k}) = \frac{1}{2} + \frac{1}{2}\mathbf{b}(\mathbf{k}) \cdot \boldsymbol{\sigma}$



$$\mathcal{O}(\mathbf{k}) = o(\mathbf{k}) + o(\mathbf{k}) \cdot \boldsymbol{\sigma}$$



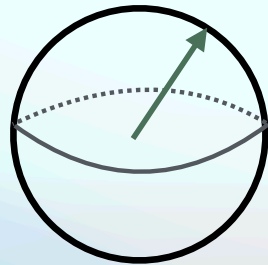
Quantum geometric nesting

Bloch vector representation

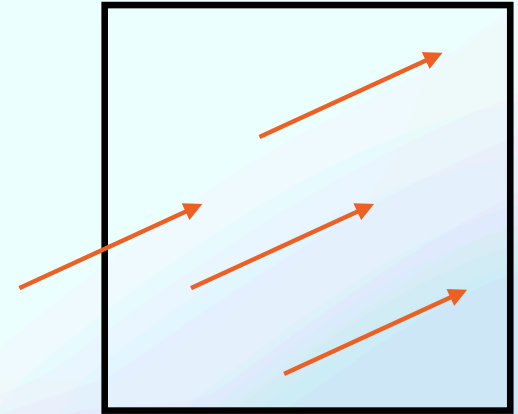
N bands, N_L flat

$$P(\mathbf{k}) = \frac{N_L}{N} + \frac{1}{2} \mathbf{b}(\mathbf{k}) \cdot \boldsymbol{\lambda}$$

$$\boldsymbol{\lambda} \in \text{SU}(N)$$



$$\mathcal{O}(\mathbf{k}) = o(\mathbf{k}) + \mathbf{o}(\mathbf{k}) \cdot \boldsymbol{\lambda}$$

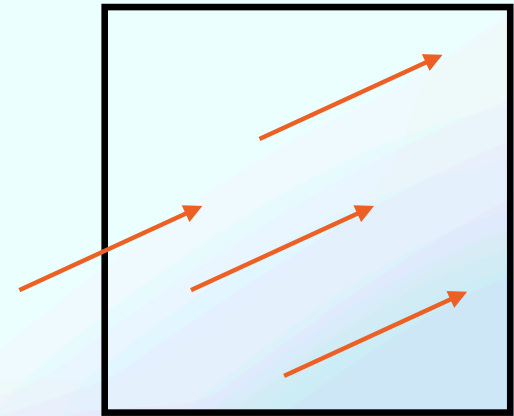


Quantum geometric nesting

$$\chi(\mathbf{Q}) = \sum_{\mathbf{k}} \frac{\text{Tr}[\mathcal{O}^\dagger(\mathbf{k})\mathcal{O}(\mathbf{k})]}{T} \left(\frac{N_L^2}{N} + \frac{1}{2} \tilde{\mathbf{b}}_o(\mathbf{k} + \mathbf{Q}) \cdot \mathbf{b}(\mathbf{k}) \right),$$

Relates quantum geometry at $\mathbf{k}$ and $\mathbf{k}+\mathbf{Q}$

$$\tilde{\mathbf{b}}_o(\mathbf{k} + \mathbf{Q}) \equiv \mathbf{b}(\mathbf{k} + \mathbf{Q}) - N [\hat{\mathbf{o}}(\mathbf{k}) \times \mathbf{b}(\mathbf{k} + \mathbf{Q}) \times \hat{\mathbf{o}}(\mathbf{k})]$$



QGN condition:

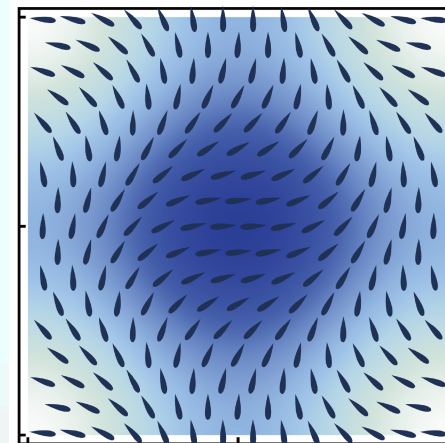
$$\tilde{\mathbf{b}}_o(\mathbf{k} + \mathbf{Q}) \parallel \mathbf{b}(\mathbf{k}), \quad \forall \mathbf{k} \in \text{BZ}$$

Example

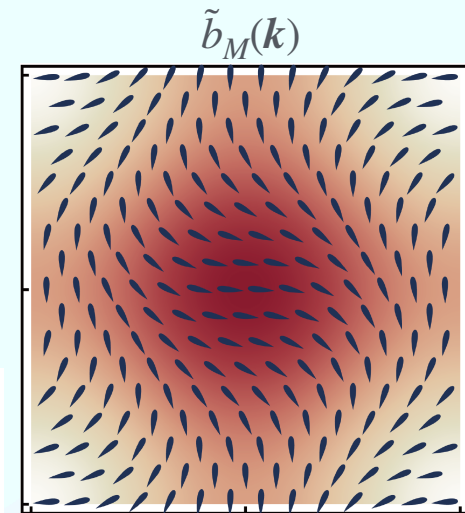
JS Hofmann et al, 2022,2023

$$H_0 = t \sum_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} \begin{pmatrix} -\mu & -ie^{i\alpha_{\mathbf{k}}^{\uparrow}} & 0 & 0 \\ ie^{-i\alpha_{\mathbf{k}}^{\uparrow}} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha_{\mathbf{k}}^{\downarrow}} \\ 0 & 0 & -ie^{i\alpha_{\mathbf{k}}^{\downarrow}} & -\mu \end{pmatrix} c_{\mathbf{k}}$$

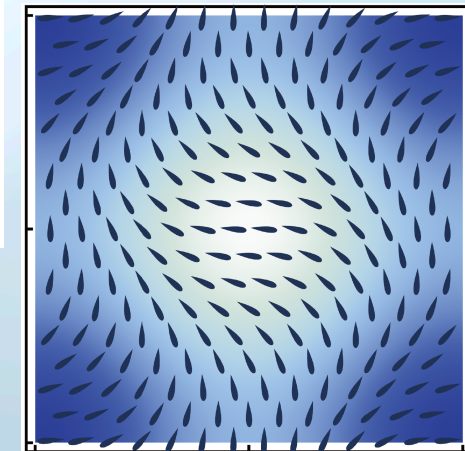
$$\alpha_{\mathbf{k}}^{\sigma} = \eta_{\sigma} (\cos k_x + \cos k_y)$$



$b(\mathbf{k})$



$\tilde{b}_N(\mathbf{k})$

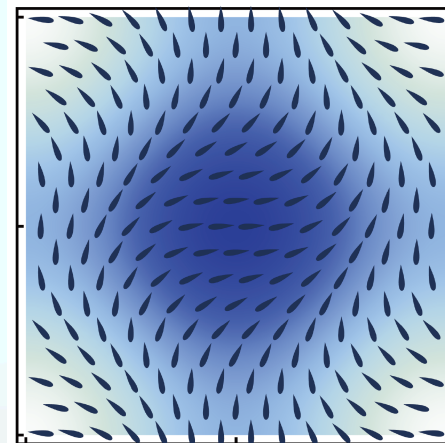


Example

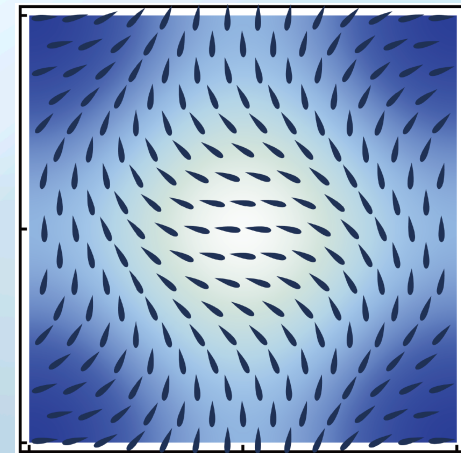
JS Hofmann et al, 2022,2023

$$H_0 = t \sum_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} \begin{pmatrix} -\mu & -ie^{i\alpha_{\mathbf{k}}^{\uparrow}} & 0 & 0 \\ ie^{-i\alpha_{\mathbf{k}}^{\uparrow}} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha_{\mathbf{k}}^{\downarrow}} \\ 0 & 0 & -ie^{i\alpha_{\mathbf{k}}^{\downarrow}} & -\mu \end{pmatrix} c_{\mathbf{k}}$$

$$\alpha_{\mathbf{k}}^{\sigma} = \eta_{\sigma} (\cos k_x + \cos k_y)$$



$b(\mathbf{k})$



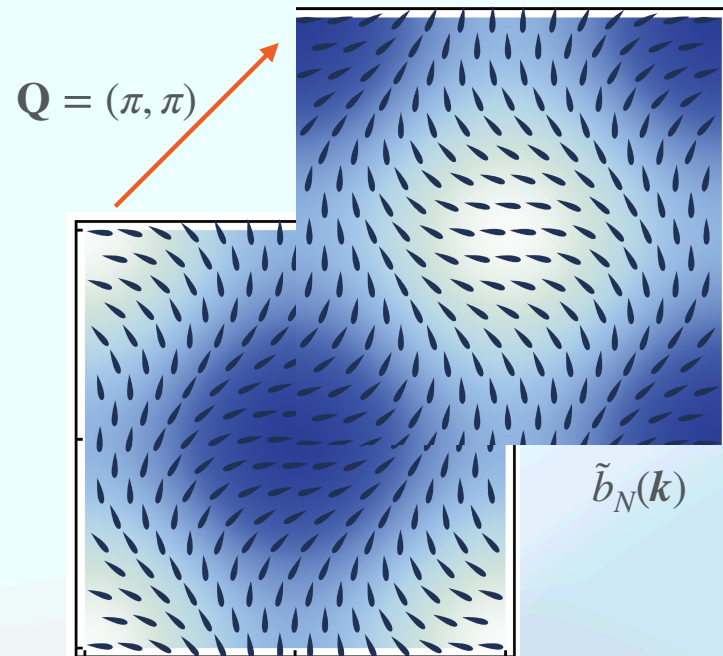
$\tilde{b}_N(\mathbf{k})$

Example

JS Hofmann et al, 2022,2023

$$H_0 = t \sum_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} \begin{pmatrix} -\mu & -ie^{i\alpha_{\mathbf{k}}^{\uparrow}} & 0 & 0 \\ ie^{-i\alpha_{\mathbf{k}}^{\uparrow}} & -\mu & 0 & 0 \\ 0 & 0 & -\mu & ie^{-i\alpha_{\mathbf{k}}^{\downarrow}} \\ 0 & 0 & -ie^{i\alpha_{\mathbf{k}}^{\downarrow}} & -\mu \end{pmatrix} c_{\mathbf{k}}$$

$$\alpha_{\mathbf{k}}^{\sigma} = \eta_{\sigma} (\cos k_x + \cos k_y)$$



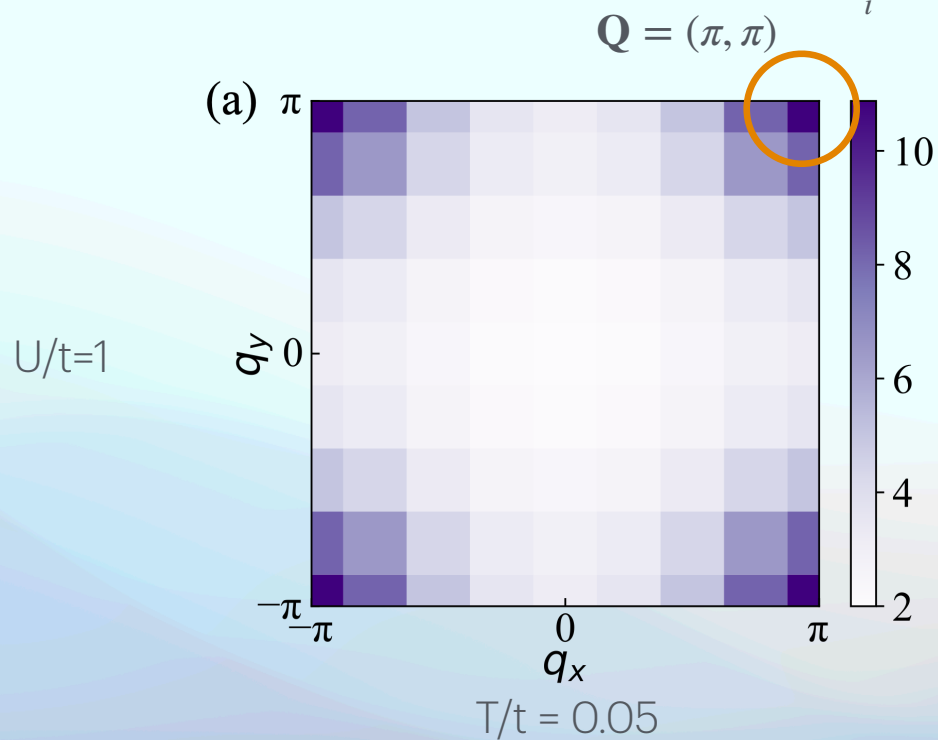
$b(\mathbf{k})$

QGN!

QMC Simulation

Include Hubbard interaction $H = H_0 + U \sum_i (n_{i,A}^2 + n_{i,B}^2)$

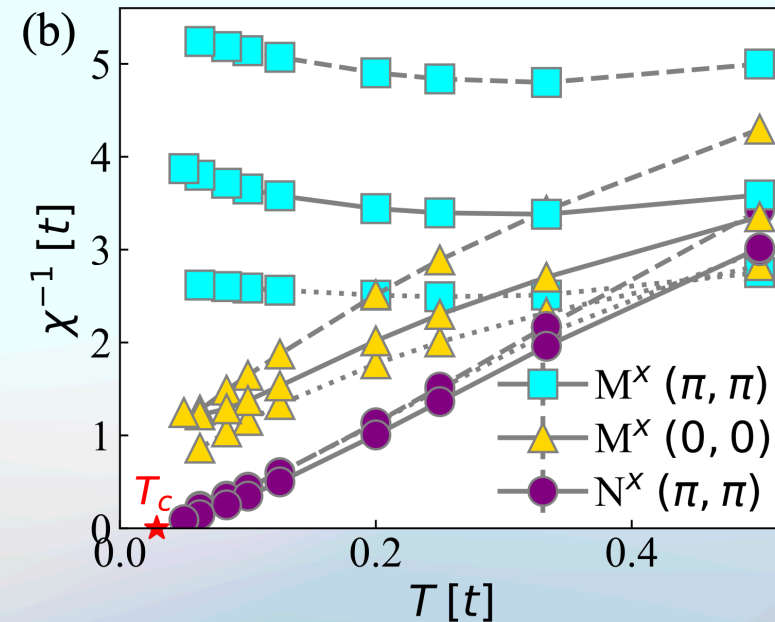
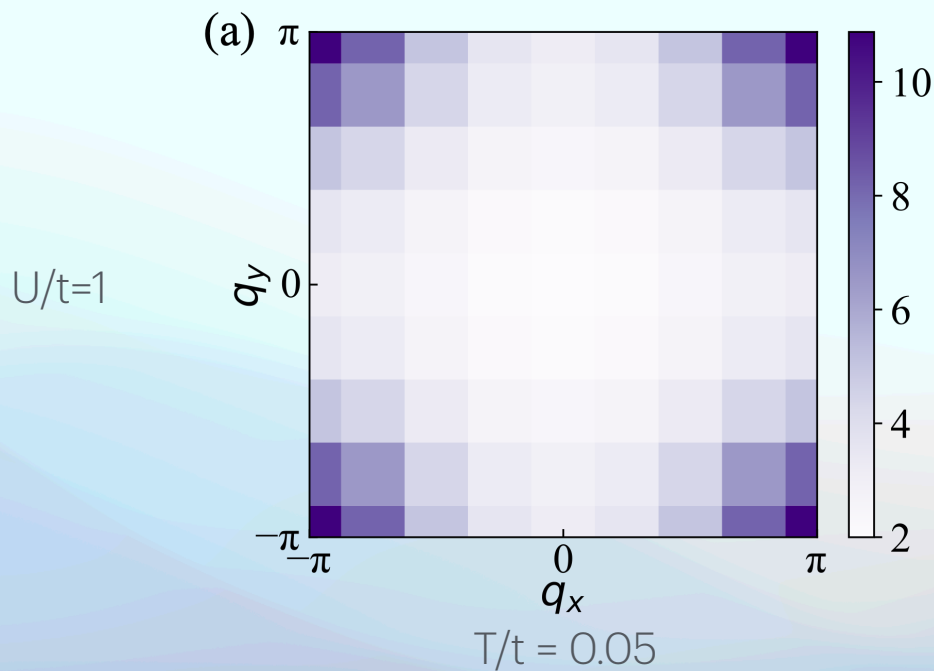
Use dQMC for 8x8 lattice



QMC Simulation

Include Hubbard interaction $H = H_0 + U \sum_i (n_{i,A}^2 + n_{i,B}^2)$

Use dQMC for 8x8 lattice



Superconductivity

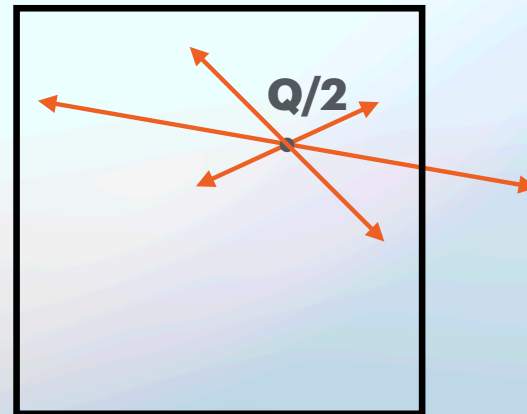
With attractive interactions in the flat band, can get exotic superconductivity

$$\chi_Q^{\text{pp}} = \frac{1}{4T} \frac{|\nu|}{\text{arctanh}|\nu|} \sum_{\mathbf{k}} o_0^2(\mathbf{k}) \left(Nn^2 + \frac{1}{2} \mathbf{b}^R(\mathbf{k} + \mathbf{Q}) \cdot \mathbf{b}(-\mathbf{k}) \right)$$

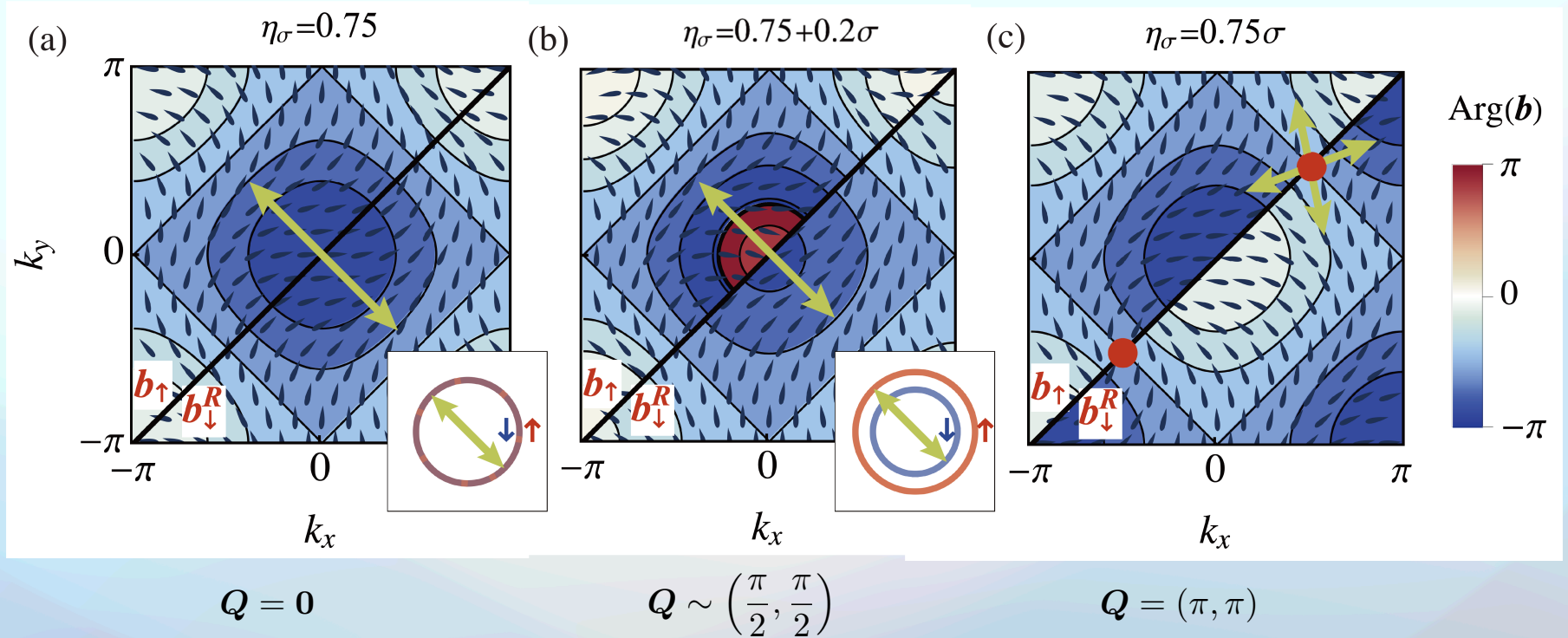
Similar nesting conditions

$$\mathbf{b}^R(\mathbf{k} + \mathbf{Q}) \parallel \mathbf{b}(-\mathbf{k})$$

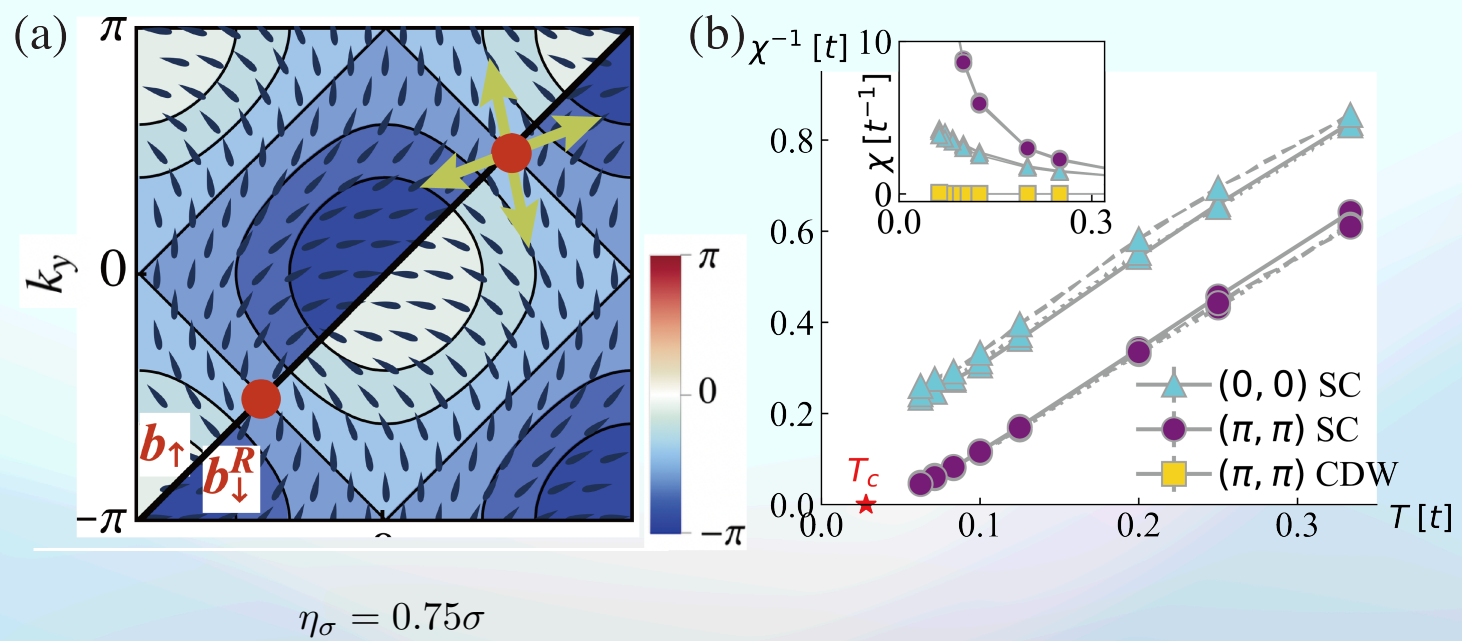
$$\mathbf{b}^R\left(\frac{\mathbf{Q}}{2} + \frac{\mathbf{k}}{2}\right) \parallel \mathbf{b}\left(\frac{\mathbf{Q}}{2} - \frac{\mathbf{k}}{2}\right)$$



Superconducting QGN



QMC

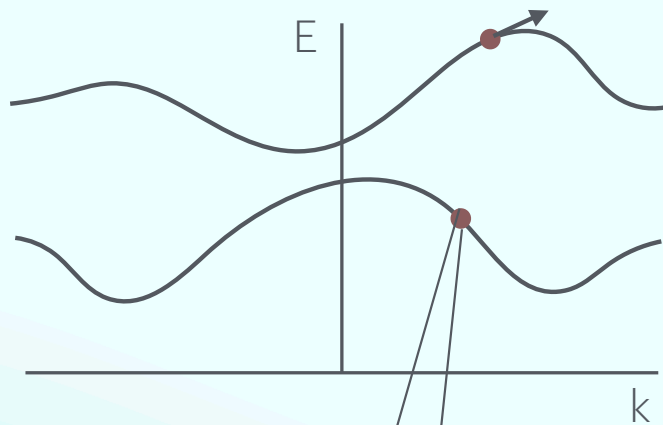


Summary

- We developed an algebraic approach to quantum geometric nesting
- It allows intuitive study of ordering instabilities of flat bands
- QMC validates the approach for a simple model system
- We also obtain relations between a “high temperature stiffness” and a generalized quantum metric (for both particle/hole and superconducting orders at any $\mathbf{Q}$)
- ?? Is there a way to include *non-symmetry breaking “orders”* like FQAHE?

Kinetic theory

Bands



$|\psi_{nk}\rangle$

Geometry

$$\epsilon_n(k) \rightarrow \mathbf{v}_n(k)$$

Energetics

$$\sigma_{xx} \sim \langle v_n^x v_n^x \rangle$$

$$\Omega_{n,\alpha\beta}(k) \quad \text{Berry curvature}$$

$$g_{n,\alpha\beta}(k) \quad \text{Quantum metric}$$

$$\sigma_{xy} \sim \langle \Omega \rangle$$

Souza, Wilkens, Martin 2000
Various suggestions

Role in interacting systems?

Linear response

A host of “conventional” (but still complex) transport coefficients

$$\begin{aligned} \mathbf{j} &= \mathbf{L}^{11} \mathbf{E} + \mathbf{L}^{12} (-\nabla T), \\ \mathbf{j}^q &= \mathbf{L}^{21} \mathbf{E} + \mathbf{L}^{22} (-\nabla T), \end{aligned}$$

Ashcroft+Mermin

And some originating from Berry curvature

Anomalous Hall effect

$$\mathbf{j}_e^{ah} = e^2 \sum_n \int \frac{d^d \mathbf{k}}{(2\pi)^d} n_F(\omega_n(\mathbf{k})) \mathbf{\Omega}_n(\mathbf{k}) \times \mathbf{E}$$

Karplus+Luttinger, 1954!

Gyromagnetic effect

$$\alpha_{ij}^{\text{GME}} = \frac{i\omega\tau e}{1 - i\omega\tau} \sum_n \int [d\mathbf{k}] (\partial f / \partial \epsilon_{\mathbf{k}n}) v_{\mathbf{k}n,i} m_{\mathbf{k}n,j}$$

Ma+Pesin 2015; Song et al 2016

Chiral anomaly...

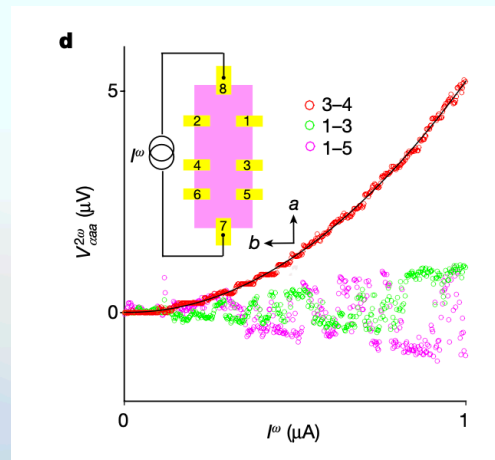
Non-linear response

Non-linear Hall effect

$$j_a^0 = \chi_{abc} \mathcal{E}_b \mathcal{E}_c^*, \quad j_a^{2\omega} = \chi_{abc} \mathcal{E}_b \mathcal{E}_c,$$

$$\chi_{abc} = -\varepsilon_{adc} \frac{e^3 \tau}{2(1 + i\omega\tau)} \int_k f_0(\partial_b \Omega_d)$$

Sodemann+Fu, 2015



Q. Ma *et al*, 2019 (WTe2)

Non-linear response

Quantum metric effects

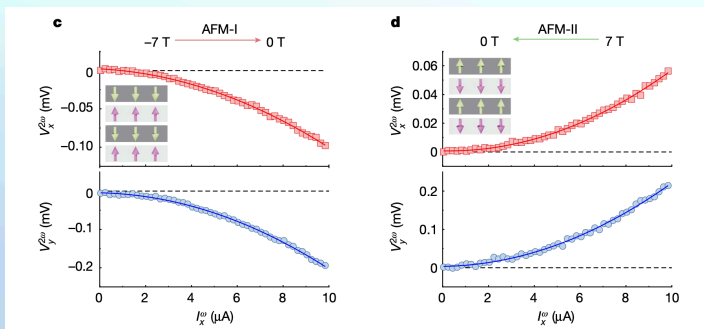
$$\sigma_{a;bc}^{\text{BCPD}} = \frac{e^3}{\hbar} \sum_{m,p,k} \int [dk] f_m [\partial_a \tilde{\mathcal{G}}_{mp}^{bc} + \partial_b \tilde{\mathcal{G}}_{mp}^{ac} + \partial_c \tilde{\mathcal{G}}_{mp}^{ab}].$$

“Band-resolved quantum metric dipole”

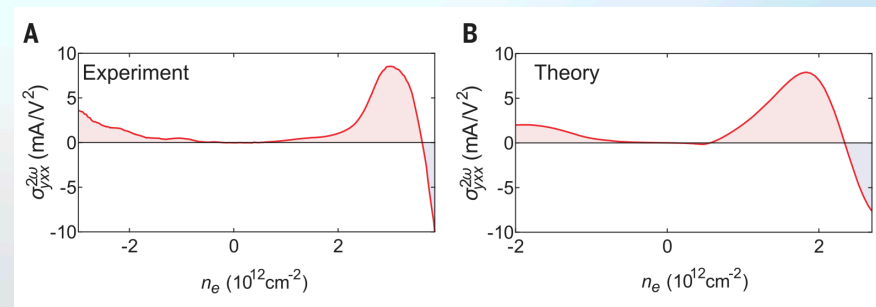
K. Das et al, 2023

D. Kaplan et al, 2024

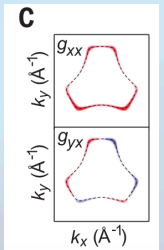
MnBi₂Te₄



N. Wang et al, 2023



A. Gao et al, 2023



Response to field gradient

$$j_{\text{geom.}}^{\mu}(\mathbf{r}) = -\frac{Q^2}{2\hbar} \int \frac{d^D \mathbf{K}}{(2\pi)^D} f_0(\mathbf{K}) E_{\nu\lambda}^{(0)} \frac{\partial g^{\nu\lambda}(\mathbf{K})}{\partial K_{\mu}},$$

Quantum metric dipole

M. Lapa + T. Hughes, 2019

$$E_{\mu\nu} = \partial E_{\mu} / \partial x_{\nu}$$

Theoretical approaches

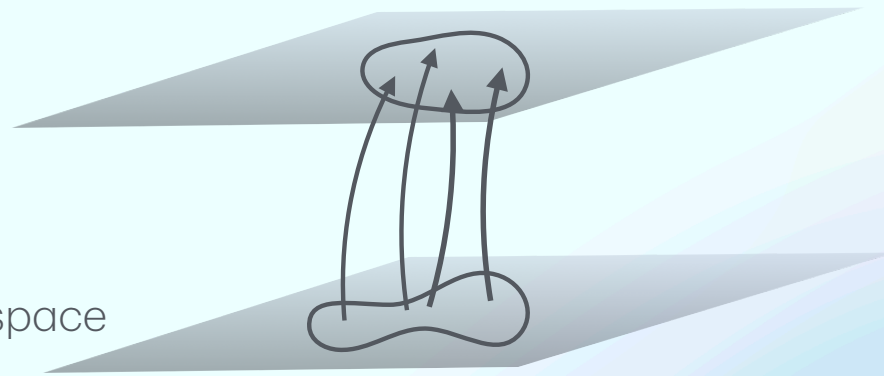
- Quantum response theory (~Kubo)
 - Usually regarded as most rigorous method
 - Formal. Simple results often obtained by very technical route.
 - Does not show dis-equilibrium
- Semi-classics
 - Intuitive
 - Directly reveals deviations from equilibrium
 - Approximate?

Semiclassics

- Boltzmann equation

$$\partial_t f_n + \frac{dx_\mu}{dt} \frac{\partial f_n}{\partial x_\mu} + \frac{dk_\mu}{dt} \frac{\partial f_n}{\partial k_\mu} = \mathcal{C}[f]$$

Incompressible flow of states in phase space



- One particle equations of motion

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{v}_n = \nabla_{\mathbf{k}} \tilde{\epsilon}_{n\mathbf{k}} - \frac{d\mathbf{k}}{dt} \times \boldsymbol{\Omega}_{n\mathbf{k}}, \\ \frac{d\mathbf{k}}{dt} &= \mathbf{F}_n = -e\mathbf{E} - e \frac{d\mathbf{x}}{dt} \times \mathbf{B}. \end{aligned}$$

The power of semiclassics

- Anomalous Hall effect

$$\mathbf{j}_{ah} = \int_k f_n e \mathbf{v}_{ah} = -e \int_k f_n \frac{d\mathbf{k}}{dt} \times \boldsymbol{\Omega}_n = e^2 \mathbf{E} \times \int_k f_n \boldsymbol{\Omega}_n$$

Immediately gives AHE

- Thermal conductivity

$$f_n = n_F(\epsilon; T(x)) + g_n$$

$$g_n = \tau n'_F \mathbf{v}_n \cdot \frac{(\epsilon - \mu) \nabla T}{T^2}$$

n.b. essential to include $T(x)$.

$$j_\mu^q = \int_k (\epsilon_n - \mu) v_n^\mu g_n = -\frac{\partial_\nu T}{T^2} \tau \int_k n'_F (\epsilon - \mu)^2 v_n^\mu v_n^\nu$$

“Standard” thermal conductivity

The power of semiclassics

- Anomalous Hall effect

$$\mathbf{j}_{ah} = \int_k f_n e \mathbf{v}_{ah} = -e \int_k f_n \frac{d\mathbf{k}}{dt} \times \boldsymbol{\Omega}_n = e^2 \mathbf{E} \times \int_k f_n \boldsymbol{\Omega}_n$$

Immediately gives AHE

- Thermal conductivity

$$f_n = n_F(\epsilon; T(x))$$

These and many other results agree with Kubo exactly. This is because the semiclassical approximation is exact to first order in spatial gradients.

$$g_n = \tau n'_F \mathbf{v}_n \cdot \frac{(\epsilon - \mu) \nabla T}{T^2}$$

n.b. essential to include $T(x)$.

$$j_\mu^q = \int_k (\epsilon_n - \mu) v_n^\mu g_n = -\frac{\partial_\nu T}{T^2} \tau \int_k n'_F (\epsilon - \mu)^2 v_n^\mu v_n^\nu$$

“Standard” thermal conductivity

Semiclassics

- Boltzmann equation

$$\partial_t f_n + \frac{dx_\mu}{dt} \frac{\partial f_n}{\partial x_\mu} + \frac{dk_\mu}{dt} \frac{\partial f_n}{\partial k_\mu} = \mathcal{C}[f]$$

Incompressible flow of states in phase space

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$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{v}_n = \nabla_{\mathbf{k}} \tilde{\epsilon}_{n\mathbf{k}} - \frac{d\mathbf{k}}{dt} \times \boldsymbol{\Omega}_{n\mathbf{k}}, \\ \frac{d\mathbf{k}}{dt} &= \mathbf{F}_n = -e\mathbf{E} - e \frac{d\mathbf{x}}{dt} \times \mathbf{B}. \end{aligned}$$

Q. Niu ++

Just written down! $i\hbar\partial_t\rho = [H, \rho]$

Wavepacket
approximation

$$i\hbar\partial_t\mathbf{x} = [H, \mathbf{x}]$$

Semiclassics

- Boltzmann equation

$$\partial_t f_n + \frac{dx_\mu}{dt} \frac{\partial f_n}{\partial x_\mu} + \frac{dk_\mu}{dt} \frac{\partial f_n}{\partial k_\mu} = \mathcal{C}[f]$$

Incompressible flow of states in phase space

- One particle equations of motion

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{v}_n = \nabla_{\mathbf{k}} \tilde{\epsilon}_{n\mathbf{k}} - \frac{d\mathbf{k}}{dt} \times \boldsymbol{\Omega}_{n\mathbf{k}}, \\ \frac{d\mathbf{k}}{dt} &= \mathbf{F}_n = -e\mathbf{E} - e \frac{d\mathbf{x}}{dt} \times \mathbf{B}. \end{aligned}$$

Q. Niu ++

Just written down! $i\hbar\partial_t\rho = [H, \rho]$

Two semiclassical approximations at the same time

Wavepacket approximation

$$i\hbar\partial_t\mathbf{x} = [H, \mathbf{x}]$$

Kinetics

Liouville-von Neumann equation $i\hbar\partial_t\rho = [H, \rho]$

Systematic derivation of Boltzmann-like equations

Intuitively $f_n(k) \sim \langle n, k | \rho | n, k \rangle$

Two issues:

- Track spatial dependence
- Off-diagonal terms in density matrix (inter-band coherence)

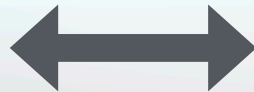
Wigner transformation

$$F(k, X) = \int dx e^{ikx} F\left(X + \frac{x}{2}, X - \frac{x}{2}\right) \left\langle X + \frac{x}{2} \left| \rho \right| X - \frac{x}{2} \right\rangle$$

Wigner-Weyl quantization

$$C = AB$$

Operators



$$C(k, X) = A(k, X) \star B(k, X)$$

Functions on phase space

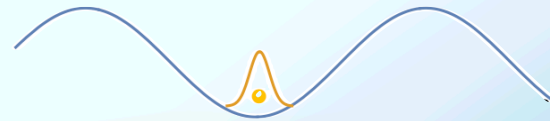
Moyal/star product

$$\star = \exp \left(\frac{i\hbar}{2} \left(\vec{\nabla}_x \cdot \vec{\nabla}_p - \vec{\nabla}_p \cdot \vec{\nabla}_x \right) \right) = \exp \left(\frac{i\hbar}{2} \epsilon^{\alpha\beta} \vec{\partial}_\alpha \vec{\partial}_\beta \right)$$

Symplectic form $[q_\alpha, q_\beta] = i\hbar \epsilon^{\alpha\beta}$

Semi-classical expansion

$$\star = \exp \left(\frac{i\hbar}{2} \epsilon^{\alpha\beta} \vec{\partial}_\alpha \vec{\partial}_\beta \right) = 1 + \frac{i\hbar}{2} \epsilon^{\alpha\beta} \vec{\partial}_\alpha \vec{\partial}_\beta + \dots$$



Slow spatial variation

Systematic procedure to carry out gradient expansion while working in both position and momentum.

Star diagonalization

Even with phase space formalism, $\mathbf{H}, \mathbf{F}$, etc are still matrices

To pass to a semi-classical kinetic equation, we want to eliminate off-diagonal terms.

Diagonalization is compatible with star product

$$\mathbf{H} = U \star \tilde{h} \star U^\dagger \quad \mathbf{F} = U \star \tilde{f} \star U^\dagger$$

Almost the desired diagonal Hamiltonian and distribution

Formal diagonalization carried out order by order in gradient expansion.

BUT we need to ensure gauge invariance.

Gauge invariance

Quantum mechanics allows an arbitrary phase for each eigenfunction. Within a band this is the usual origin of Berry gauge field.

$$U \rightarrow U \star E^{i\theta(x,k)}$$

$$\tilde{f} \rightarrow E^{-i\theta} \star \tilde{f} \star E^{i\theta}$$

Transforms non-trivially
because of star product

Gauge invariant form

$$f_n = \text{tr} \left[U \star p_n \tilde{f} \star U^\dagger \right]$$

Projector on nth band

Energy density

$$\rho_n^E = \frac{1}{2} \text{tr} [U \star p_n (\tilde{h} \star \tilde{f} + \tilde{f} \star \tilde{h}) \star U^\dagger]$$

Gauge invariance

We worked these out to second order in gradients:

$$f_n = \tilde{f}_n + \hbar \varepsilon_{\alpha\beta} \partial_\alpha (\tilde{f}_n A_{n\beta}) + \frac{\hbar^2}{2} \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \partial_{\alpha\sigma}^2 (\tilde{f}_n (\Lambda_\beta \Lambda_\lambda)_{nn})$$

$$h_n \equiv \tilde{h}_n + \hbar \varepsilon_{\alpha\beta} \partial_\alpha \tilde{h}_n (A_{n\beta} + \hbar \varepsilon_{\sigma\lambda} A_{n\lambda} \partial_\sigma A_{n\beta}) - \hbar^2 \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \partial_{\alpha\sigma}^2 \tilde{h}_n \left(\frac{1}{4} \{ \Lambda_\beta, \Lambda_\lambda \}_{nn} - A_{n\beta} A_{n\lambda} \right)$$

$$\Lambda_\alpha = -iU^\dagger \star \partial_\alpha U$$

$$A_\alpha = \text{diag}(\Lambda_\alpha) \quad \text{Berry gauge field}$$

Now simply

$$\mathcal{N} = \int_{x,p} \sum_n f_n$$

f_n is electron density in a band

$$E = \int_{x,p} \sum_n f_n h_n$$

$f_n h_n$ is energy density in a band

Kinetic equation

$$\partial_t f_n + \partial_\alpha \mathcal{J}_\alpha = 0$$

Just continuity equation for density in a band

c.f. simple convection

$$\mathcal{J}_\alpha = f_n v_\alpha$$



Incompressible flow

Boltzmann equation

Here, to second order:

$$\begin{aligned} \mathcal{J}_\alpha = & \varepsilon_{\alpha\beta} \operatorname{tr} \left[f \left(\partial_\beta h + \hbar (\varepsilon_{\sigma\lambda} \partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}) \left(1 - \frac{\hbar}{2} \Omega_{\mu\nu} \right) + \frac{\hbar^2}{2} \varepsilon_{\sigma\lambda} \varepsilon_{\mu\nu} \partial_{\sigma\mu}^2 h \partial_\beta g_{\nu\lambda} \right) \right] \\ & + \hbar^2 \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \partial_\sigma \operatorname{tr} \left[f \left(\frac{1}{2} \varepsilon_{\mu\nu} \partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2} \partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu} \partial_{\mu\beta}^2 h g_{\nu\lambda} \right) \right] - \frac{\hbar^2}{24} \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \varepsilon_{\mu\nu} \partial_{\sigma\mu}^2 \operatorname{tr} [f \partial_{\beta\lambda\nu}^3 h] \end{aligned}$$

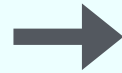
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quantum metric $g_{\alpha\beta} = \frac{1}{2} \operatorname{diag} (\Lambda_\alpha \Lambda_\beta + \Lambda_\beta \Lambda_\alpha) - A_\alpha A_\beta = \frac{1}{2} \operatorname{tr} (\partial_\alpha P_n \partial_\beta P_n)$

Kinetic equation

$$\begin{aligned} \mathcal{J}_\alpha = & \epsilon_{\alpha\beta} \operatorname{tr} \left[f \left(\partial_\beta h + \hbar (\epsilon_{\sigma\lambda} \partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}) \left(1 - \frac{\hbar}{2} \Omega_{\mu\nu} \right) + \frac{\hbar^2}{2} \epsilon_{\sigma\lambda} \epsilon_{\mu\nu} \partial_{\sigma\mu}^2 h \partial_\beta g_{\nu\lambda} \right) \right] \\ & + \hbar^2 \epsilon_{\alpha\beta} \epsilon_{\sigma\lambda} \partial_\sigma \operatorname{tr} \left[f \left(\frac{1}{2} \epsilon_{\mu\nu} \partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2} \partial_t g_{\beta\lambda} + \epsilon_{\mu\nu} \partial_{\mu\beta}^2 h g_{\nu\lambda} \right) \right] - \frac{\hbar^2}{24} \epsilon_{\alpha\beta} \epsilon_{\sigma\lambda} \epsilon_{\mu\nu} \partial_{\sigma\mu}^2 \operatorname{tr} [f \partial_{\beta\lambda\nu}^3 h] \end{aligned}$$

- General and exact up to second order in semiclassical expansion
- Explicitly contains real space and momentum space, and capable of describing inhomogeneous systems. Magnetic field can be included by modifying $\epsilon_{\alpha\beta}$
- Quantum geometry of bands appears explicitly, takes local band Hamiltonian as input
- Only band-intrinsic quantities appear: but these are bands renormalized by quantum corrections.
- Ready to attack all sorts of problems!

Example: separable problem

$$\mathbf{H}(x, p) = \mathbf{H}_0(p) - eV(x)\mathbf{I}_N$$

Band form Diagonal potential

Kinetic equation (relaxation time approx)

$$\begin{aligned} \partial_t f = & -\partial_{x_i} \left[f \left(v_i + \frac{e}{\hbar} \Omega_{ij} \partial_{x_j} V + e \tau_{ij} \partial_{x_j}^2 V + \frac{e^2}{\hbar} \left(\partial_{k_j} \tau_{il} - \frac{1}{2} \partial_{k_i} \tau_{jl} \right) \partial_{x_j} V \partial_{x_l} V + \frac{e}{\hbar} \left(\hbar v_j \tau_{il} - \frac{1}{2} \partial_{k_i} g_{jl} \right) \partial_{x_j x_l}^2 V \right) \right] \\ & - \frac{e}{\hbar} \partial_{k_i} f \partial_{x_i} V + \frac{e}{2\hbar} \partial_{x_i x_l}^2 (f \partial_{k_j} g_{il} \partial_{x_j} V) - \frac{e}{\hbar} \partial_{k_i x_j}^2 (f g_{jl} \partial_{x_i x_l}^2 V) + \frac{1}{24\hbar} \partial_{x_i x_j x_l}^3 f \partial_{k_i k_j k_l}^3 \varepsilon \\ & + \frac{e}{24\hbar} \partial_{k_i k_j k_l}^3 f \partial_{x_i x_j x_l}^3 V + \mathcal{I}[f] + \mathcal{O}(\partial_x^4). \end{aligned} \quad (3.29)$$

$$T_{n;\mu\nu} = - \sum_{m \neq n} \frac{\langle \partial_{p_\mu} u_n | u_m \rangle \langle u_m | \partial_{p_\mu} u_n \rangle + (\mu \leftrightarrow \nu)}{\epsilon_n - \epsilon_m} \quad \begin{array}{l} \text{"band-normalized quantum metric"} \\ \text{(not a purely geometric quantity)} \end{array}$$

Non-equilibrium current

Linear conductivity: q dependence $\sigma_{ij}^{(1)}(q, \omega) = \sigma_{ij}^{(1,0)}(\omega) + \sigma_{ijl}^{(1,1)}(\omega) i q_l + \mathcal{O}(q^2)$

$$\sigma_{ijl}^{(1,1)}(\omega) = \sum_{n=1}^N e^2 \int_k \left(f'_n \frac{v_i v_j v_l}{(i\omega - \tau^{-1})^2} + f_n \frac{1}{2} (v_j \mathsf{T}_{li} + v_l \mathsf{T}_{ij} - \hbar^{-1} \partial_{k_j} \mathsf{g}_{li} - \hbar^{-1} \partial_{k_l} \mathsf{g}_{ij}) - f_n \frac{\hbar^2}{2} (M_{j;il} + M_{l;ij}) \right)$$

Lapa+Hughes

Quadratic conductivity

$$\sigma_{ijl}^{(2)}(q, \omega; q', \omega') = \frac{e^3}{\hbar^2} \sum_{n=1}^N \int_k f_n \left[\frac{\partial_{k_i k_j k_l}^3 \varepsilon}{(i\omega - \tau^{-1})(i\omega' - \tau^{-1})} - \left(\frac{\hbar v_i}{2} \mathsf{T}_{jl} - \partial_{k_i} \mathsf{T}_{jl} + \partial_{k_j} \mathsf{T}_{il} \right) + \frac{\partial_{k_j} \Omega_{ij}}{i\omega' - \tau^{-1}} \right] + \mathcal{O}(q)$$

Band-normalized quantum metric dipole

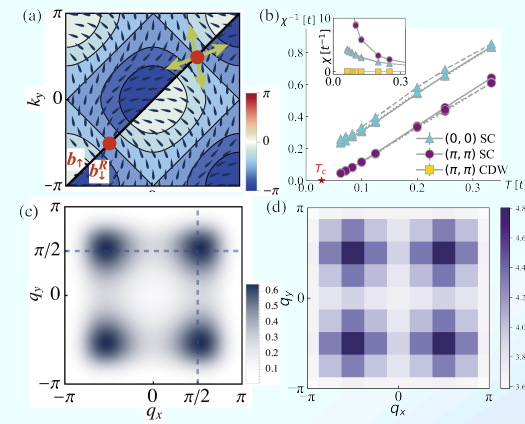
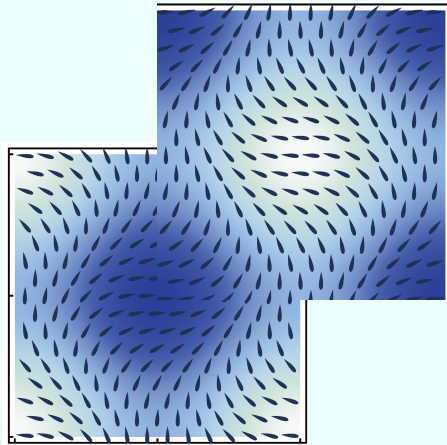
Sodemann+Fu

Summary

$$\begin{aligned} \mathcal{J}_\alpha = & \varepsilon_{\alpha\beta} \operatorname{tr} \left[f \left(\partial_\beta h + \hbar (\varepsilon_{\sigma\lambda} \partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}) \left(1 - \frac{\hbar}{2} \Omega_{\mu\nu} \right) + \frac{\hbar^2}{2} \varepsilon_{\sigma\lambda} \varepsilon_{\mu\nu} \partial_{\sigma\mu}^2 h \partial_\beta g_{\nu\lambda} \right) \right] \\ & + \hbar^2 \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \partial_\sigma \operatorname{tr} \left[f \left(\frac{1}{2} \varepsilon_{\mu\nu} \partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2} \partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu} \partial_{\mu\beta}^2 h g_{\nu\lambda} \right) \right] - \frac{\hbar^2}{24} \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \varepsilon_{\mu\nu} \partial_{\sigma\mu}^2 \operatorname{tr} [f \partial_{\beta\lambda\nu}^3 h] \end{aligned}$$

- We found a procedure to obtain semiclassical electron dynamics organized by spatial gradients
- We found a general result for the phase space current to *second* order in gradients, capable of handling inhomogeneous systems and magnetic fields
- Prior results for linear and non-linear response are recovered, and in some cases corrected
- Still needed: a scattering/interactions theory with the same validity

Thanks



$$\mathcal{J}_\alpha = \varepsilon_{\alpha\beta} \text{tr} \left[f \left(\partial_\beta h + \hbar (\varepsilon_{\sigma\lambda} \partial_\sigma h \Omega_{\lambda\beta} - \Omega_{t\beta}) \left(1 - \frac{\hbar}{2} \Omega_{\mu\nu} \right) + \frac{\hbar^2}{2} \varepsilon_{\sigma\lambda} \varepsilon_{\mu\nu} \partial_\sigma^2 h \partial_\beta g_{\nu\lambda} \right) \right] \\ + \hbar^2 \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \partial_\sigma \text{tr} \left[f \left(\frac{1}{2} \varepsilon_{\mu\nu} \partial_\mu h \partial_\nu g_{\beta\lambda} - \frac{1}{2} \partial_t g_{\beta\lambda} + \varepsilon_{\mu\nu} \partial_\mu^2 h g_{\nu\lambda} \right) \right] - \frac{\hbar^2}{24} \varepsilon_{\alpha\beta} \varepsilon_{\sigma\lambda} \varepsilon_{\mu\nu} \partial_\sigma^2 \text{tr} [f \partial_{\beta\lambda\nu}^3 h]$$

